155 Lecture 5

Bring colored chalk!

Def: If λ a partition, the conjugate $\lambda' = (\lambda_1', \dots, \lambda_m')$ is defined by λ_i' = length of i^{th} column of λ.
Equivalently, this is transpose of λ.

Recall RSK...

Ex: $\pi = 4236517$

	P	Q
P_1	$\begin{array}{c} 4 \end{array}$	$\begin{array}{c} 1 \end{array}$
P_2	$\begin{array}{c} 2 \\ 4 \end{array}$	$\begin{array}{c} 1 \\ 2 \end{array}$
P_3	$\begin{array}{cc} 2 & 3 \\ 4 & \end{array}$	$\begin{array}{cc} 1 & 3 \\ 2 & \end{array}$
P_4	$\begin{array}{ccc} 2 & 3 & 6 \\ 4 & & \end{array}$	$\begin{array}{ccc} 1 & 3 & 4 \\ 2 & & \end{array}$
P_5	$\begin{array}{ccc} 2 & 3 & 5 \\ 4 & 6 & \end{array}$	$\begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & \end{array}$
P_6	$\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & & \end{array}$	$\begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 & \\ 6 & & \end{array}$
P_7	$\begin{array}{cccc} 1 & 3 & 5 & 7 \\ 2 & 6 & & \\ 4 & & & \end{array}$	$\begin{array}{cccc} 1 & 3 & 4 & 7 \\ 2 & 5 & & \\ 6 & & & \end{array}$

Observation: As we go from P_i to P_{i+1}, the value in a given box will never increase.

Theorem: (Schensted) Consider $\pi \in S_n$. The length of the longest increasing subsequence of π = length of first row of $P(\pi)$! Length of longest decreasing " " = length of first column of $P(\pi)$.

To prove this, we need some Lemmas.

Lemma 1: If $\pi = x_1 x_2 \dots x_n$ and x_k enters P_{k-1} in column j, then the longest increasing subsequence of π ending in x_k has length j.

Proof: Induction on k. Trivial for $k=1$. Suppose it holds for values up to $k-1$.

Need to show existence of such a increasing subsequence of π. Let y be elemt of P_{k-1} in box $(1, j-1)$. Then $y < x_k$ since x_k enters in column j. By induction, $\exists$ increasing subsequence σ of π ending in y of length $j-1$, so since $y < x_k$ and x_k is inserted after y, $\sigma\, x_k$ is an increasing sequence of length j.

Need to show there is not a longer increasing subsequence $\tilde{\sigma}$ ending in x_k. Suppose there is and let x_i be entry preceding x_k in $\tilde{\sigma}$. So $x_i < x_k$. By induction, since x_i ends an increasing subsequence of length $\geq j$, in RSK, x_i enters in a column $\geq j$. So the element y in cell $(1,j)$ of P_i satisfies

$y \leq x_i$ which is $< x_k$. $\Rightarrow$ $y < x_k$.
But by Obs., we know that an entry of a given box never increases w/ subsequent insertion. $\Rightarrow\Leftarrow$

Cor: The longest increasing subsequence in π is the length of the first row in $P(\pi)$.

To prove the other part of Schensted's Theorem, we need:

Prop (Schensted): Let π^r be the reverse of π. If $P(\pi) = P$, then $P(\pi^r) = P^t$, where t denotes transposition.

Then to finish Schensted's Theorem,
length of 1^{st} column of $P(\pi)$ =
length of 1^{st} row of $P(\pi^r)$ =
length of longest increasing subsequence of π^r =
" " " " decreasing " " of π.

So we just need to prove Prop. How?
Note that we can define column insertion the same way we defined row insertion, just replacing row by column in the def.

If column insertion of x into P gives P', write $c_x(P) = P'$.
" row " " ", $r_x(P) = P'$

Ex: If we use $\pi = 4236517$ & perform column insertion, we get

$$4 \longrightarrow 2\,4 \longrightarrow \begin{array}{ll} 2 & 4 \\ 3 & \end{array} \longrightarrow \begin{array}{ll} 2 & 4 \\ 3 & \\ 6 & \end{array} \longrightarrow \begin{array}{ll} 2 & 4 \\ 3 & 6 \\ 5 & \end{array} \longrightarrow \begin{array}{lll} 1 & 2 & 4 \\ 3 & 6 & \\ 5 & & \end{array} \longrightarrow \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 6 & \\ \cline{1-2} 5 & & \\ \cline{1-1} 7 & & \\ \cline{1-1} \end{array}$$

Observation: This is the transpose of what we got earlier from row insertion. $\begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 7 \\ \hline 2 & 6 & & \\ \cline{1-2} 4 & & & \\ \cline{1-1} \end{array}$ (clear from the definition)

Lemma: For any partial tableau P and distinct elements $x, y \notin P$, $c_y r_x(P) = r_x c_y(P)$.

Proof: Lots of cases – can read in Sagan.

Now we can finally prove 2

Prop (Schensted): Let π^r be the reverse of π, If $P(\pi) = P$, then $P(\pi^r) = P^t$, where t denotes transposition.

Proof: $P(\pi^r) = r_{x_1} \cdots r_{x_{n-1}} r_{x_n}(\emptyset)$ (by def.)

$= r_{x_1} \cdots r_{x_{n-1}} c_{x_n}(\emptyset)$ (since initial tableau is $\emptyset$)

$= c_{x_n} r_{x_1} \cdots r_{x_{n-1}}(\emptyset)$ (by Lemma)

$\vdots$

$= c_{x_n} c_{x_{n-1}} \cdots c_{x_1}(\emptyset)$ (by induction)

$= P^t$ (def. of column insertion)

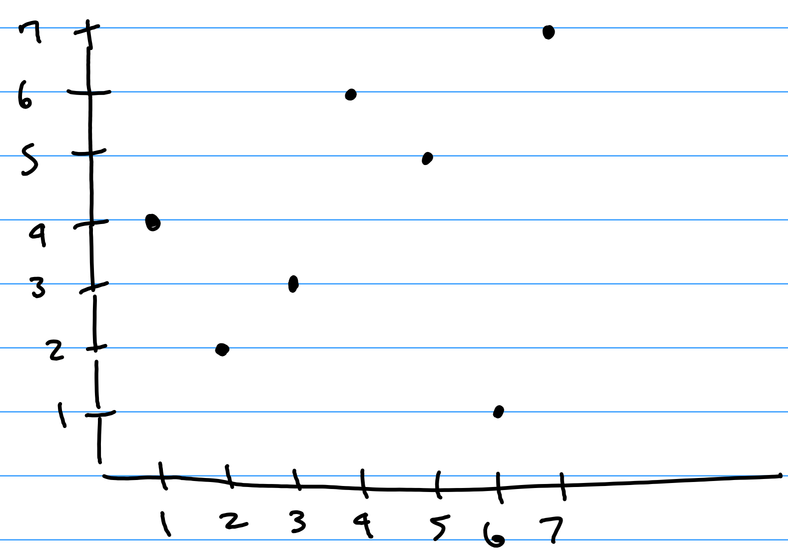
This completes the proof of Schensted's Theorem.

<u>Theorem</u>: (Schutzenberger) $\pi \xrightarrow{RS} (P,Q) \Rightarrow$
$\pi^{-1} \xrightarrow{RS} (Q,P)$

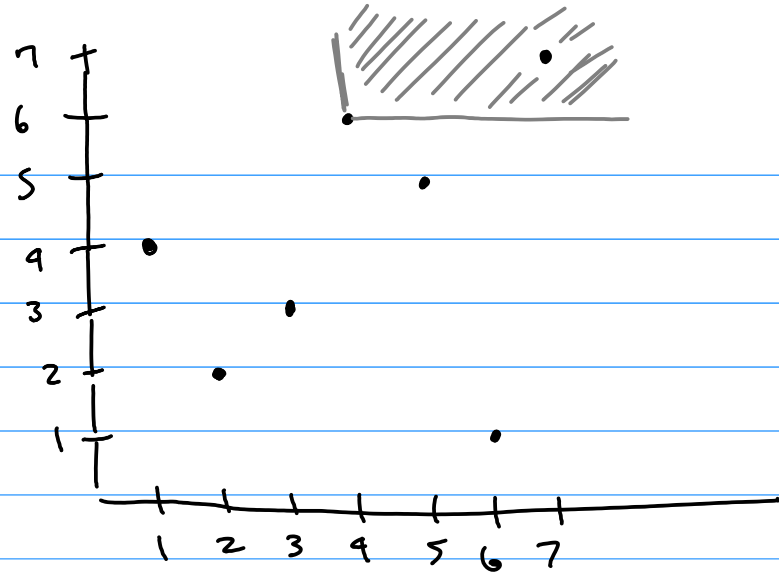
We will prove this by using another interpret. of RSK — Viennot's shadow line construction.

First, graph perm $\pi = x_1 \ldots x_n$ in the Cartesian plane, putting a point w/ coord's (i, x_i) $\forall i$.

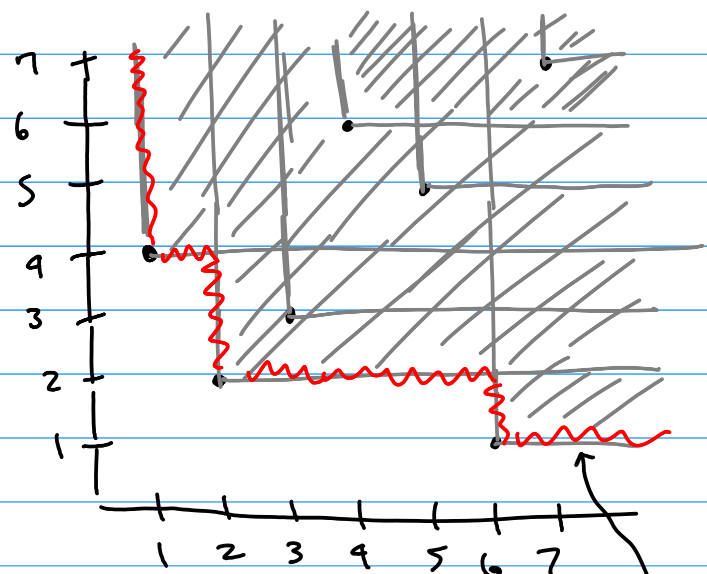
Ex: 4236517 $\rightsquigarrow$



Imagine light shining from origin so that each box casts shadow w/ boundaries $\parallel$ to coord. axes.



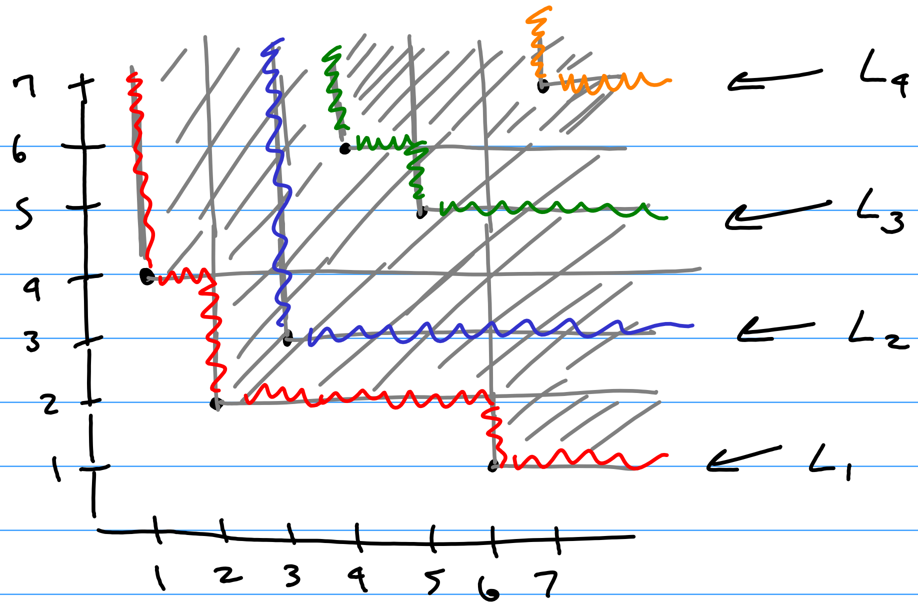
eg here is the shadow cast by (4,6)



Do this for all points. Consider the pts that are not in shadow of another point. The <u>first shadow line</u> = boundary of combined shadows of these boxes.

So that is this bdy

To form the second shadow line, remove the points on first shadow line + repeat process.



Define x_{L_i} = the x-coordinate of L_i's vertical ray and

y_{L_i} = the y-coordinate of L_i's horiz. ray

The shadow lines comprise the <u>shadow diagram</u> of π.

Recall that we had $P(\pi) = \begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 6 & & \\ 4 & & & \end{matrix}$

and $Q(\pi) = \begin{matrix} 1 & 3 & 4 & 7 \\ 2 & 5 & & \\ 6 & & & \end{matrix}$

ASK! Compare $P(\pi)$ + $Q(\pi)$ to the shadow lines — what can we say ?!

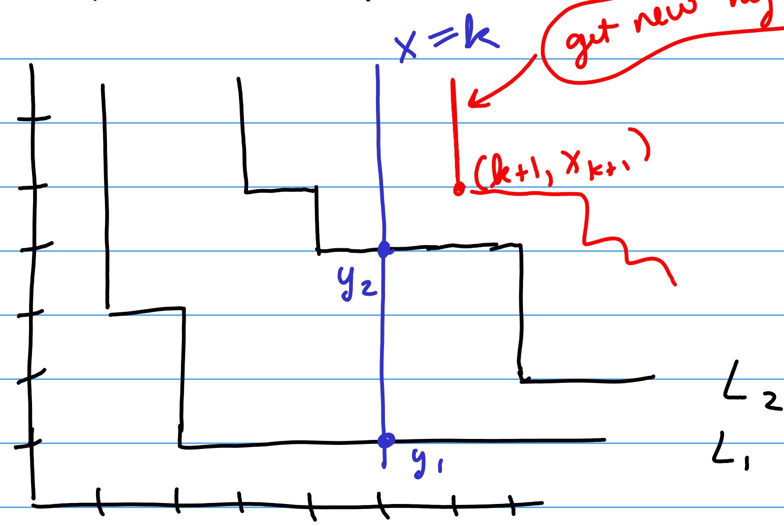
It appears that $P_{1j} = y_{L_j}$ and $Q_{1j} = x_{L_j}$

In fact: the boxes on line L_j are precisely those elemts passing through the $(1,j)$ cell during construction of P . . .

(write out the P_j again . . .)

Lemma 2: Let the shadow diagram of $\pi = x_1 \dots x_n$ be constructed as before. Suppose the vert. line $x=k$ intersects i of the shadow lines. Let y_j be the y-coord of the lowest point of intersection w/ L_j. Then the first row of P_k is $y_1\, y_2 \dots y_i$.

Proof: Induction on k, trivial for $k=0$. Assume result holds for $x=k$ & consider the line $x=k+1$.



Induction: First row of P_k is $y_1 \underset{y_i}{\underset{\shortparallel}{y_2}}$

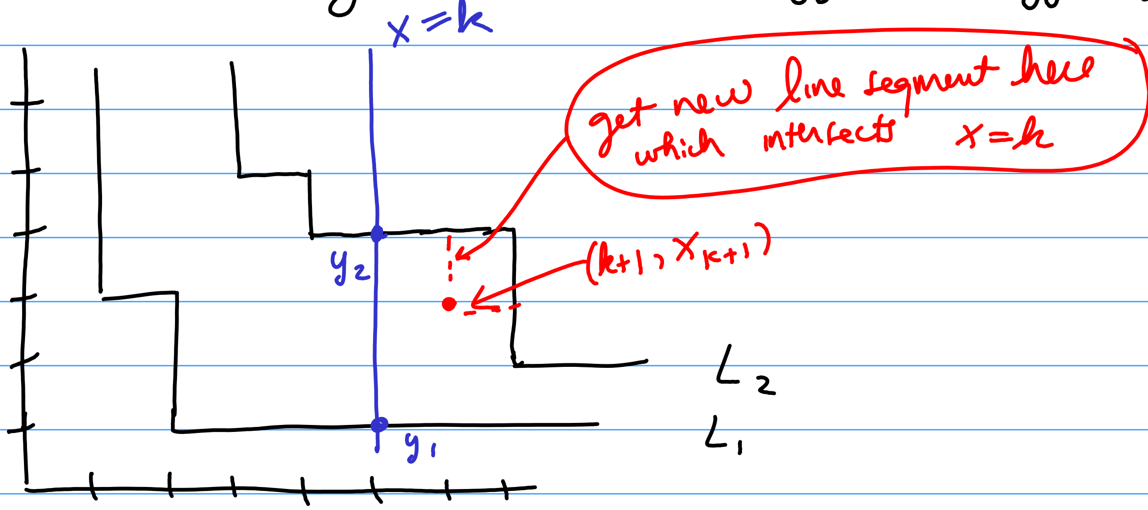
Case 1: $x_{k+1} > y_i$

Then the point $(k+1, x_{k+1})$ starts a new shadow line. So considering intersection of $x=k+1$ w/ shadow lines, the values of $y_1, \dots, y_i$ don't change & there is a new intersection $y_{i+1} = x_{k+1}$.

What happens in RSK when x_{k+1} is inserted? Since first row was $y_1 \dots y_i$, and $y_i < x_{k+1}$, x_{k+1} just goes to end of row. This is what the

Lemma predicts.

Case 2: $x_{k+1} < y_i$. So $y_1 < \dots < y_{j-1} < x_{k+1} < y_j < \dots < y_i$



In this case, $(k+1, x_{k+1})$ gets added to the shadow line L_j. So intersecting w/ $x = k+1$, the lowest coord of L_j is $y_j' = x_{k+1}$, & all other y-values stay the same. What happens in RSK when we add x_{k+1} to a tableau w/ first row $y_1 y_2 \dots y_{j-1} y_j \dots y_i$?
x_{k+1} replaces y_j. This is what the Lemma predicts. ∎

From proof we see we can read shadow diagram L to R to see what happens in construction of $P(\pi)$.

Obs: At time k, the line $x=k$ intersects one shadow line in a ray or line segmt & all the others in single pts.
Ray $\Leftrightarrow$ placing elemt at end of 1st row
Segmt $\Leftrightarrow$ bumping an elemt

All other intersection pts are unchanged in 1^{st} row

<u>Cor</u>: If $\pi \xrightarrow{RS} (P, Q)$ and π has shadow lines L_j, then $\forall j$, $P_{1j} = y_{L_j}$ and $Q_{1j} = x_{L_j}$.

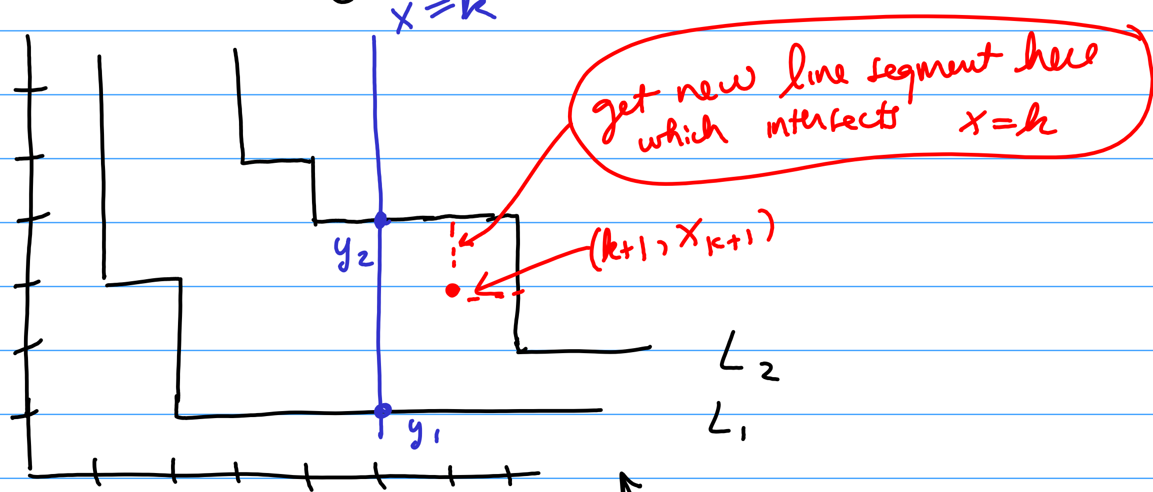
<u>Pf</u>: Statement for P comes from Lemma 2.
For Q: we add entry k to Q in cell $(1, j)$ when $x_k >$ all elements in first row of P_{k-1}. By obs., this happens precisely when line $x = k$ intersects shadow line L_j in a vert ray — i.e. $y_{L_j} = k = Q_{1j}$ as desired.

How to recover the other rows of P & Q from shadow diagram?

ASK! <u>Question</u>: What in the shadow diagram corresponds to numbers inserted into 2^{nd} row of P?

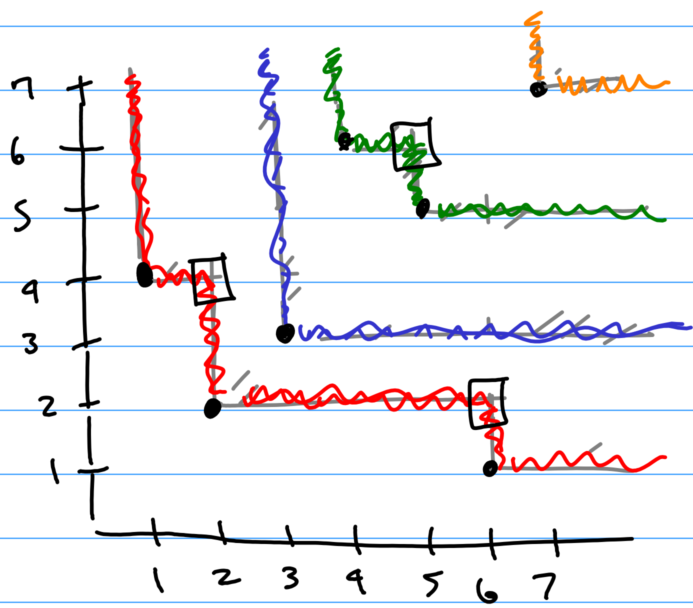
Recall Case 2 of Lemma ...

Case 2: $x_{k+1} < y_i$. So $y_1 < \dots < y_{j-1} < x_{k+1} < y_j < \dots < y_i$



When we add x_{k+1} to first row $y_1 \dots y_i$, it bumps y_j. Picture is

$\therefore$ the elements inserted into 2nd row are the NE corners of shadow lines: represented by $\square$ below



So we can get 2nd rows of P & Q by doing shadow construction w/ these points, & continue to get P & Q

<u>Def:</u> The i^{th} skeleton of $\pi \in S_n$, $\pi^{(i)}$, is defined inductively by $\pi^{(1)} = \pi$ and

$$\pi^{(i)} = \begin{matrix} k_1 & k_2 & & k_m \\ \downarrow & \downarrow & \cdots & \downarrow \\ l_1 & l_2 & & l_m \end{matrix}$$ where

(k_j, l_j) are the NE corners of the shadow diagram of $\pi^{(i-1)}$ listed in lex order.
Let $L_j^{(i)}$ denote the shadow lines for $\pi^{(i)}$

This discussion proves the following:

<u>Theorem: 1</u> Suppose $\pi \xrightarrow{RS} (P,Q)$. Then $\pi^{(i)}$ is a partial perm s.t. $\pi^{(i)} \xrightarrow{RS} (P^{(i)}, Q^{(i)})$ where $P^{(i)}$ consists of rows i & below of P & sim. for Q. Further,

$$P_{ij} = y_{L_j^{(i)}} \quad \text{and} \quad Q_{ij} = x_{L_j^{(i)}}$$

Now note that taking the inverse of π corresponds to reflecting shadow diagram in line $y=x$. $\therefore$ by Theorem 1, we have Schützenberger's Theorem:

$$\pi \xrightarrow{RS} (P,Q) \implies \pi^{-1} \xrightarrow{RS} (Q,P)$$ $\blacksquare$

Knuth Equivalence + Green's Theorem

<u>Def:</u> $\pi, \sigma \in S_n$ are said to be P-equivalent, written $\pi \overset{P}{\cong} \sigma$, if $P(\pi) = P(\sigma)$.

On S_3 we have these equiv. classes:

$\{123\}$ $\quad\quad$ $\{213, 231\}$ $\quad\quad$ $\{132, 312\}$ $\quad\quad$ $\{321\}$

notice

123 $\quad\quad$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\quad\quad$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad\quad$ $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$

Need an alternative def. of P-equivalence.

<u>Def:</u> Suppose $x < y < z$. Then $\pi, \sigma \in S_n$ differ by a Knuth relation of the first kind, written $\pi \overset{1}{\cong} \sigma$ if

1. $\pi = x_1 \ldots yxz \ldots x_n$ and $\sigma = x_1 \ldots yzx \ldots x_n$ or vice-versa

They differ by a Knuth relation of the second kind, written $\pi \overset{2}{\cong} \sigma$ if

2. $\pi = x_1 \ldots xzy \ldots x_n$ and $\sigma = x_1 \ldots zxy \ldots x_n$ or vice-versa.

The two perms are called <u>Knuth equivalent</u>, written $\pi \overset{K}{\cong} \sigma$, if one can get from π to σ using a sequence of Knuth relations.

Looking at example, the Knuth-equiv class + P-equiv classes coincide.

Theorem (Knuth): If $\pi, \sigma \in S_n$ then
$\pi \overset{K}{\cong} \sigma$ iff $\pi \overset{P}{\cong} \sigma$

Proof: $(\Longrightarrow)$ Enough to show $\pi \overset{P}{\cong} \sigma$ whenever π + σ differs by a single Knuth relation. In fact, result for $\overset{2}{\cong}$ follows from result for $\overset{1}{\cong}$:

$$\begin{aligned} \pi \overset{2}{\cong} \sigma \Rightarrow \ & \pi^r \overset{1}{\cong} \sigma^r && \text{(by def.)} \\ \Rightarrow \ & P(\pi^r) = P(\sigma^r) && \text{(result for } \overset{1}{\cong}) \\ \Rightarrow \ & P(\pi)^t = P(\sigma)^t \\ \Rightarrow \ & P(\pi) = P(\sigma). \end{aligned}$$

Now assume $\pi \overset{1}{\cong} \sigma$.

$\pi = x_1 \ldots yxz \ldots x_n$ and $\sigma = x_1 \ldots yzx \ldots x_n$

Let P be the tableau obtained by inserting the elements before y.

Need to show: $r_z r_x r_y(P) = r_x r_z r_y(P)$.

Idea: the insertion path of y creates a "barrier" s.t. paths for x and z lie to the left + right of this barrier, no matter in which order they are inserted. So since the paths don't intersect, $r_x r_z$ + $r_z r_x$ have same effect.

Induction on # of rows of P. If P has no rows, then $P(zxy) = \begin{matrix} x & y \\ z & \end{matrix} = P(xyz)$