155 Lecture 4

Today: prove that the Specht modules are irred.

Given a numbering T of λ, recall row gp $R(T)$ & column gp $C(T)$.

Let $A = \mathbb{C}[S_n]$ be gp ring of S_n.
Given a numbering T of a diagram w/ n boxes (where $\{1,\dots,n\}$ each occur once), define elemts in A:

$$a_T = \sum_{p \in R(T)} p \quad \in A$$

$$b_T = \sum_{q \in C(T)} \operatorname{sgn}(q)\, q \quad \in A$$

Recall column word of a tableau:

$$\begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 6 & 8 \\ \hline 7 \\ \cline{1-1} \end{array} \rightsquigarrow 7\ 2\ 1\ 6\ 4\ 8\ 5$$

Total order on numberings w/ n boxes: $T' > T$ if:
(1) shape of T larger than shape of T in lex order
or (2) T' and T have same shape, & largest differing entry occurs earlier in column word of T'

Tabloid = equiv. class of numberings:

$$\begin{array}{|ccc|} \hline 1 & 4 & 7 \\ \hline 3 & 6 \\ \cline{1-2} 2 & 5 \\ \cline{1-2} \end{array} = \begin{array}{|ccc|} \hline 4 & 7 & 1 \\ \hline 6 & 3 \\ \cline{1-2} 2 & 5 \\ \cline{1-2} \end{array}$$

Recall: Exercise: $b_T \cdot b_T = \# C(T)\, b_T$

M^λ = v.s. w/ basis the tabloids $\{T\}$ of shape λ.
S_n acts on M^λ.

For each numbering T of λ define $v_T \in M^\lambda$ by

$$v_T = b_T \cdot \{T\} = \sum_{q \in C(T)} \operatorname{sgn}(q) \{q \cdot T\}$$

Specht module S^λ :=
subspace of M^λ spanned by the v_T
(as T varies over numberings of λ)

Lemma: Let T & T' be numberings of shapes λ and λ', & assume λ does not strictly dominate λ'. If there is a pair of integers in the same row of T' & same column of T then $b_T \cdot \{T'\} = 0$. If there is no such pair, $b_T \cdot \{T'\} = \pm v_T$.

Cor: If T and T' are std tableaux w/ $T' > T$ then $b_T \cdot \{T'\} = 0$

Pf: Use Corollary (*)

Theorem: The S^λ are distinct irreps of S_n.

Proof: No $v_T = 0$ so $S^\lambda \neq 0$.

Let T be any numbering of λ.

Claim 1: $b_T \cdot M^\lambda = \mathbb{C} \cdot v_T \neq 0$.

we have $=$ because for any tabloid T', Lemma ‼

$\Rightarrow$ $b_T \cdot \{T'\} = 0$ or $b_T \cdot \{T'\} = \pm v_T$

Claim 2: $b_T \cdot M^\lambda = b_T \cdot S^\lambda$

Why? Clearly $b_T \cdot S^\lambda \subset b_T \cdot M^\lambda$. Want $\supset$

For any $\{T'\}$ of shape λ, if $b_T \cdot \{T'\} \neq 0$ then $b_T \cdot \{T'\} = \pm v_T$ and

v_T in span of $b_T \cdot v_T = b_T \cdot b_T \cdot \{T\} = \# C(T) \cdot b_T \{T\}$ by exercise

$\underbrace{\phantom{b_T\{T\}}}_{v_T}$

so each $b_T \cdot \{T'\} \subset b_T \cdot S^\lambda \Rightarrow$

$b_T \cdot M^\lambda \subset b_T \cdot S^\lambda$

$\therefore$ $b_T \cdot S^\lambda = \mathbb{C} \cdot v_T \neq 0$.

lex order

Claim 3: $b_T \cdot S^{\lambda'} = 0$ if $\lambda' > \lambda$

Why? $\lambda' > \lambda \Rightarrow$ for any T' of shape λ', $T' > T \Rightarrow$

by Corollary $\clubsuit$, $b_T \cdot \{T'\} = 0$.

Thus $b_T \cdot M^{\lambda'} = 0 \Rightarrow b_T \cdot S^{\lambda'} = 0$.

Now assume $S^\lambda = V \oplus W$ not irred. ASK!

Then $\mathbb{C} \cdot v_T = b_T \cdot S^\lambda = b_T \cdot (V \oplus W) = b_T \cdot V \oplus b_T \cdot W$.

$\mathbb{C} \cdot v_T$ is one-dim $\Rightarrow$

WLOG $\mathbb{C} \cdot v_T = b_T \cdot V \Rightarrow$

$v_T \in b_T \cdot V \subset V \Rightarrow v_T \in V$.

Note: $S^\lambda = A \cdot v_T$ for any numbering T of λ.
Because each $\sigma \in S_n$ is in A and
$\sigma \cdot v_T = v_{\sigma \cdot T}$ by exercise 3 so $A \cdot v_T$
includes all $v_{T'}$ for all numberings T' of λ.

So now $v_T \in V \Rightarrow A \cdot v_T \in V$ (V an A-module)
$\Rightarrow S^\lambda \subset V \implies S^\lambda = V. \Rightarrow W = 0$.
$\therefore$ S^λ is irreducible!

Note that the S^λ's must be distinct since
for a numbering T of λ,
$$b_T \cdot S^\lambda \neq 0 \quad \text{but} \quad b_T \cdot S^{\lambda'} = 0$$
(i.e. A acts differently on S^λ and $S^{\lambda'}$). ∎

We have found distinct reps for each $\lambda \vdash n$
so we have found all irreps of S_n.

Recall: $f^\lambda = \#$ std tableaux of shape λ.

Theorem: $$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

We will prove this combinatorially ...

Prop: The elemts v_T, as T varies over the std tableaux on λ, form a basis for S^λ.

Proof: v_T is a linear comb. of $\{T\}$ (w/ coeff 1) and elemts of the form $\{q \cdot T\}$ for $q \in C(T)$ w/ coeffs ± 1. When T is a standard tableau, $q \cdot T < T$ in ordering defined earlier, for $q \neq e$, $q \in C(T)$.

Then the v_T are indep:
Suppose $\sum x_T v_T = 0$ where T varies over std tableaux. Look at the maximal T occurring w/ nonzero coeffs in this relation. Then T is a std tableaux, $\{T\}$ occurs w/ coeff 1 in v_T, & since $T >$ all other std tableaux T' in sum, T cannot appear as a coeff in any other $v_{T'}$ (all such coeffs $q \cdot T'$ are $< T' < T$).

$\therefore \dim S^\lambda \geq f^\lambda$.

But now since we know that
$$n! = \sum_{\lambda \vdash n} (\dim S^\lambda)^2 \geq \sum_{\lambda \vdash n} (f^\lambda)^2 = n!,$$
we must have equality above, i.e. $\dim S^\lambda = f^\lambda \Rightarrow$ the $\{v_T \mid T \text{ a std tab. of shape } \lambda\}$ form a basis of S^λ. ∎

Ea Ea

Now: The RSK Algorithm

(Robinson, Schensted, & Knuth)

<u>Theorem</u>: $\exists$ bijection between $\pi \in S$ and pairs of std tableaux of same shape $\lambda \vdash n$.

$$\therefore \quad n! = \sum_{\lambda \vdash n} (f^{\lambda})^2$$

Elementary operation: the insertion of an element k into a tableau T.

1. Replace in row 1 of T the smallest element a which is larger than k by the new element k.
   (a gets "bumped" by k)
   If all elements in row 1 are $< k$, set k at the end of row 1 & stop.

2. If a has been "bumped" by k, insert a into row 2 according to rule (1) and continue.

Denote new tableau by $T \leftarrow k$

Insert 4 into

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 7 & 8 \\ \cline{1-2} \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} 1 & 3 & 4 \\ 2 & 5 \\ 6 & 8 \\ 7 \end{array}$$

The Algorithm: We associate to $\pi \in S_n$ a pair (P, Q) of SST's by inserting elements step by step.

At first: $P(0) = Q(0) = \emptyset$.
Suppose we have constructed $(P(t), Q(t))$. Then.

A) Let $P(t+1) := P(t) \leftarrow \pi(t+1)$
B) Construct $Q(t+1)$ from $Q(t)$ by putting $t+1$ into the unique position s.t. $Q(t+1)$ and $P(t+1)$ have the same shape.

Eg 14278365

$$\begin{array}{l} 1 \end{array} \qquad \begin{array}{l} 1 \end{array}$$

$$\begin{array}{l} 14 \end{array} \qquad \begin{array}{l} 12 \end{array}$$

$$\begin{array}{l} 12 \\ 4 \end{array} \qquad \begin{array}{l} 12 \\ 3 \end{array}$$

$$\begin{array}{l} 127 \\ 4 \end{array} \qquad \begin{array}{l} 124 \\ 3 \end{array}$$

$$\begin{array}{l} 1278 \\ 4 \end{array} \qquad \begin{array}{l} 1245 \\ 3 \end{array}$$

$$\begin{array}{l} 1238 \\ 47 \end{array} \qquad \begin{array}{l} 1245 \\ 36 \end{array}$$

$$\begin{array}{l} 1236 \\ 478 \end{array} \qquad \begin{array}{l} 1245 \\ 367 \end{array}$$

$$\begin{array}{l} 1235 \\ 468 \\ 7 \end{array} \qquad \begin{array}{l} 1245 \\ 367 \\ 8 \end{array}$$

<u>Theorem:</u> The RSK algorithm gives a bijection between $\pi \in S_n$ and ordered pairs $(P = P(\pi), Q = Q(\pi))$ of std tableaux of same shape.

In fact the theorem generalizes to semistd tableaux. (contribution of Knuth)

Semistd tableaux: a numbering T of λ s.t.
rows <u>weakly</u> increase +
columns strictly increase.

Note that we can generalize insertion $T \leftarrow k$ to allow for repeated entries.

1. Replace in row 1 of T the smallest element a which is larger than k by the new element k.
(a gets "bumped" by k)
If all elements in row 1 are $\leq k$, set k at the end of row 1 + stop.

w/ std tableaux, we got correspondence of perms + pairs of std tableaux.

Now will get corresp. of integer matrices & pairs of semistd tableaux

Ex: Insert 3 into the tableau T, where T is:

$$\begin{array}{llllll} 1&2&2&4&5&6&6 \\ 2&3&3&6&7 \\ 4&4&7&7&8 \\ 6&7 \\ 7&8 \end{array} \rightsquigarrow \begin{array}{lllllll} 1&2&2&3&5&6&6 \\ 2&3&3&4&7 \\ 4&4&6&7&8 \\ 6&7&7 \\ 7&8 \end{array}$$

insertion path

Ex: Insert 3 into $T \leftarrow 3$

$$\begin{array}{llllllll} 1&2&2&3&5&6&6 \\ 2&3&3&4&7 \\ 4&4&6&7&8 \\ 6&7&7 \\ 7&8 \end{array} \rightsquigarrow \begin{array}{llllllll} 1&2&2&3&3&6&6 \\ 2&3&3&4&5 \\ 4&4&6&7&7 \\ 6&7&7&8 \\ 7&8 \end{array}$$

Obs: By strict monotonicity on columns of T, the insertion path goes down & left but never to the right.

Lemma: If T is a SST then so is $T \leftarrow k$

Proof: The rows in $T \leftarrow k$ are obviously monotone, and rule (1) implies that the columns are strictly increasing.

Lemma: Suppose we perform $T \leftarrow k$ and then $(T \leftarrow k) \leftarrow l$ w/ $l \geq k$. Then the insertion path of l runs strictly to the right of the insertion path of k. Also, the insertion path of l does not extend below the bottom of the insertion path of k.
(Exercise) See examples we did.]

Using this procedure, we will construct from a matrix (all of whose entries are in $\mathbb{N}$) a pair of SST of the same shape — this is the RSK algorithm.

Let $A = (a_{ij})_{i,j \geq 1}$ be a matrix over $\mathbb{N}$ w/ only finitely many nonzero elements. Let $n = \sum_{i,j} a_{ij}$

We associate to A a $2 \times n$ matrix "scheme" s.t.

- Column $\binom{i}{j}$ appears a_{ij} times
- The first row is increasing
- Below the i's in the top row the j's are arranged in increasing order

Ex: $A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} \mapsto \begin{matrix} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 4 & 2 & 2 & 1 & 3 \end{matrix}$

$\downarrow$ List of columns corresponding to entries

$\begin{matrix} 1 & 3 & 2 & 2 & 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \end{matrix} \nearrow$

Clearly we can recover A from its $2\times n$ scheme. Note that in the $2\times n$ scheme,

i appears $\sum_{j\geq 1} a_{ij}$ times in the first row

j appears $\sum_{i\geq 1} a_{ij}$ times in the 2^{nd} row.

Now instead of working w/ the (infinite) matrices A, we just work w/ their 2-schemes — will *also* denote these by A.

$$A = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$$

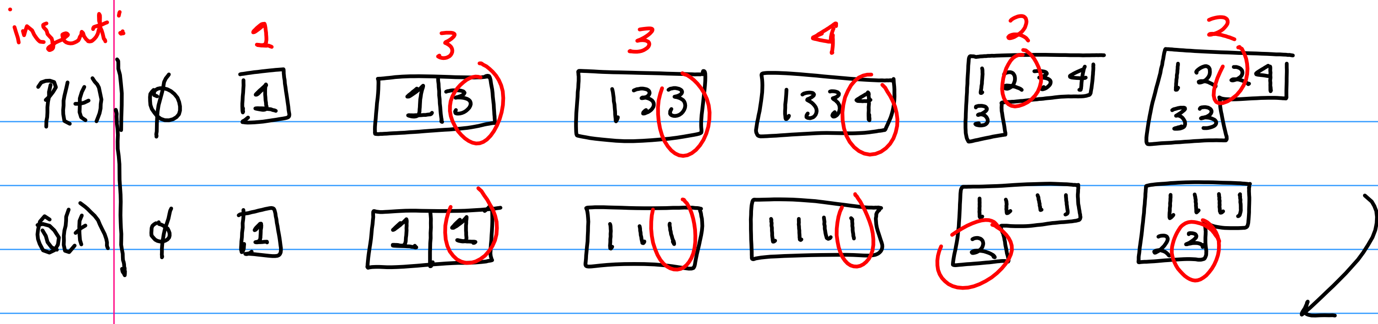
The Algorithm: We associate to A a pair (P,Q) of SST's by inserting elements step by step.

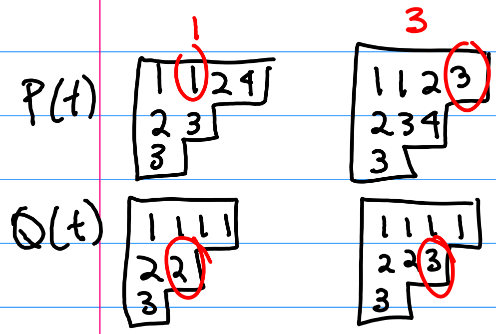
At first: $P(0) = Q(0) = \emptyset$.
Suppose we have constructed $(P(t), Q(t))$. Then.

A) Let $P(t+1) := P(t) \leftarrow j_{t+1}$

B) Construct $Q(t+1)$ from $Q(t)$ by putting i_{t+1} into the unique position s.t. $Q(t+1)$ and $P(t+1)$ have the same shape.

Example: Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 4 & 2 & 2 & 1 & 3 \end{pmatrix}$.





Point out: Q is the recording tableau: it tells us the order in which we added boxes to our partition shape. P is called insertion tableau.

Theorem: The RSK algorithm gives a bijection between the matrices A over $\mathbb{N}$ and the ordered pairs (P, Q) of SST of the same shape.

One of Schensted's motivations for RSK was study of increasing + decreasing subsequences in perms.

Def: Given $\pi = x_1 \ldots x_n \in S_n$, an increasing subsequence of π is $x_{i_1} < x_{i_2} < \ldots < x_{i_k}$

Theorem: (Schensted) Consider $\pi \in S_n$. The Length of the longest increasing subsequence of π = length of first row of $P(\pi)$! Length of longest decreasing " " = length of first column of $P(\pi)$.