155 Lecture 3

Today: construct irreps of S_n "Specht modules"

Need to define two partial orders on the set of partitions.

Lexicographic order: denoted $\mu \leq \lambda$, and means that the first i for which $\mu_i \neq \lambda_i$ (if any) has $\mu_i < \lambda_i$

Dominance order: denoted $\mu \trianglelefteq \lambda$, and means that $\mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i$ for all i.

Lexicographic order is a total order but dominance is not: $(2,2,2)$ and $(3,1,1,1)$ not comparable in dominance order

Let T and T' denote numberings of a Young diagram w/ n boxes w/ the numbers from 1 to n, w/ no repeats allowed.

S_n acts on the set of numberings: if $\sigma \in S_n$, $\sigma \cdot T$ is the numbering where $\sigma(i)$ is in the box that i was in in T.

eg. $(125)(34) \cdot \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|} \hline 5 & 4 \\ \hline 2 & 1 \\ \hline 3 \\ \cline{1-1} \end{array}$

If we fix T, there is a subgp $R(T)$ of S_n, the row gp of T, consisting of all permutations preserving the rows — (just permuting entries of each row.)

If $\lambda = (\lambda_1 \geq \dots \geq \lambda_k > 0)$ is the shape of T, $R(T)$ is a product of symm. gps $S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_k}$. Called a <u>Young subgp</u>.

Similarly there is a <u>column gp</u> $C(T)$ of perms preserving columns of T.

<u>Observe</u>: $R(\sigma . T) = \sigma R(T) \sigma^{-1}$, $C(\sigma . T) = \sigma C(T) \sigma^{-1}$

(useful later 2)

<u>Lemma</u>: Let T & T' be numberings of shapes λ & λ'. Assume λ does not strictly dominate λ'. Then exactly one of the following occurs:

(i) There are 2 distinct integers that occur in the same row of T' & same column of T

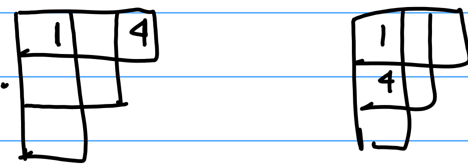
(ii) $\lambda' = \lambda$ and there is some p' in $R(T')$ & some q in $C(T)$ s.t. $p' \cdot T' = q \cdot T$

<u>Proof</u>: Suppose (i) false. Then entries of 1^{st} row of T' occur in different columns of T, so $\exists\, q_1 \in C(T)$ s.t. those entries occur in 1^{st} row of $q_1 \cdot T$.
The entries in 2^{nd} row of T' occur in different columns of T, so in diff columns of $q_1 \cdot T$, so $\exists\, q$ in $C(q_1 \cdot T) = C(T)$ not moving entries equal to those in 1^{st} row of T' which moves entries in 2^{nd} row of T' into 2^{nd} row of $q_2 \cdot$ Continue ... we get $q_1, \dots, q_k$ in $C(T)$ s.t. the entries in first k rows of T' occur in the 1^{st} k rows of $q_k \cdot q_{k-1} \cdot \dots \cdot q_1 \cdot T$. Since T and $q_k \cdot \dots \cdot q_1 \cdot T$ have same shape, $\lambda_1' + \dots + \lambda_k' \leq \lambda_1 + \dots + \lambda_k$.
True for all k so $\lambda' \trianglelefteq \lambda$.

We assumed λ doesn't strictly dominate λ' so $\lambda = \lambda'$. Let $k = \#$ of rows in λ and $q = q_k \cdot q_{k-1} \cdot \ldots \cdot q_1$. Since $\lambda = \lambda'$, $q \cdot T$ and T' have the same entries in each row so $\exists\, p' \in R(T')$ s.t. $p' \cdot T' = q \cdot T$.

Easy to see that $\underset{\text{true}}{(i)} \Rightarrow (ii)$ false

Ex: if (i) is true, eg...



T' $\qquad\qquad\qquad$ T

using $R(T')$ $\qquad$ using $C(T)$ we can't make the 2 tableaux equal! ∎

Def: The column word of a tableau is the word we get by listing the entries from bottom to top in each column, starting in left column & moving right.

Eg $\begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 & 8 \\ \hline 7 \\ \cline{1-1}\end{array} \rightsquigarrow 7\ 2\ 1\ 6\ 4\ 8\ 5$

Define a linear (= total) ordering on the set of all numberings w/ n boxes by saying $T' > T$ if:

1. shape of T' larger than shape of T in lex order

OR 2. T' and T have same shape, & the largest entry that is in a different box in the two numberings occurs earlier in column word of T' than in column word of T.

eg $\begin{array}{ccc}1&2&3\\4&5\end{array} > \begin{array}{ccc}1&2&4\\3&5\end{array} > \begin{array}{ccc}1&3&4\\2&5\end{array} > \begin{array}{ccc}1&2&5\\3&4\end{array} > \begin{array}{ccc}1&3&5\\2&4\end{array}$

41523 $\quad$ 31524 $\quad$ 21534 $\quad$ 31425 $\quad$ 21435

Obs: If T is a standard tableau, then for any $p \in R(T)$ and $q \in C(T)$,

$$p \cdot T \geq T \quad \text{and} \quad q \cdot T \leq T$$

$R(T)$ moves bigger #'s closer to beginning of column word

$C(T)$ does the opposite

(*) **Cor:** If T and T' are standard tableaux w/ $T' > T$ then there is a pair of integers in the same row of T' and the same column of T.

Proof: Since $T' > T$, the shape of T can't dominate the shape of T'. If there is no such pair then case (ii) of Lemma holds – $\exists\ p' \in R(T')$ and $q \in C(T)$ s.t. $p' \cdot T' = q \cdot T$.

By Obs., $T' \leq p' \cdot T' = q \cdot T \leq T \Rightarrow T' \leq T$.

This is a contradiction.

Now: Specht modules

Def: A tabloid is an equivalence class of numberings of a Young diagram where 2 are equiv. if corresp. rows contain same entries.

Use $\{T\}$ to denote equiv. class of T.

So $\{T'\} = \{T\}$ exactly when $T' = p \cdot T$ for $p \in R(T)$.

Sometimes drawn as:

$$\begin{array}{|ccc|}\hline 1 & 4 & 7 \\ \hline 3 & 6 & \\ \cline{1-2} 2 & 5 & \\ \cline{1-2} \end{array} = \begin{array}{|ccc|}\hline 4 & 7 & 1 \\ \hline 6 & 3 & \\ \cline{1-2} 2 & 5 & \\ \cline{1-2} \end{array}$$

S_n acts on the set of tabloids by $\sigma \cdot \{T\} = \{\sigma . T\}$

__Recall:__ the __group ring__ $\mathbb{C}[G]$ of G consist of all complex linear combinations $\sum x_\sigma \sigma$ w/ mult determined by mult in G.

Let $A = \mathbb{C}[S_n]$ be gp ring of S_n.
Given a numbering T of a diagram w/ n boxes (where $\{1, .., n\}$ each occur once), define elemts in A:

$$a_T = \sum_{p \in R(T)} p \quad \in A$$

$$b_T = \sum_{q \in C(T)} \operatorname{sgn}(q)\, q \quad \in A$$

$$c_T = a_T b_T \quad \in A$$

Called __Young symmetrizers__.

__Ex:__ $T = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline 5 & \\ \cline{1-1} \end{array}$ $\Rightarrow$ $a_T = e + (13) + (24) + (13)(24)$

$$b_T = e - (34) - (12) - (15) - (25) + (125) + (152)$$
$$+ (34)(12) + (34)(15) + (34)(25) - (34)(125) - (34)(152)$$

PSet #2

__Exercise 1:__ (a) For $p \in R(T)$ & $q \in C(T)$, show that $p \cdot a_T = a_T \cdot p = a_T$ and
$q \cdot b_T = b_T \cdot q = \operatorname{sgn}(q)\, b_T$

(b) Show that $a_T \cdot a_T = \# R(T)\, a_T$ and $b_T \cdot b_T = \# C(T)\, b_T$

__Ex:__ $\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$ $a_T = e + (12)$ so $a_T \cdot a_T = (e + (12))(e + (12))$
$= 2e + 2(12) = 2a_T$.
and $2 = \# R(T)$

Goal: To find one irrep for each conj. class in S_n.
Conj. classes $\longleftrightarrow$ partitions λ of n, where conj. class $C(\lambda)$ consists of those permutations whose cycle decomp. has the form
$$(\underset{\text{length } \lambda_1}{\uparrow})(\underset{\lambda_2}{\uparrow})\cdots(\underset{\lambda_k}{\uparrow})$$
if $\lambda = (\lambda_1, \dots, \lambda_k)$.

Exercise 2: Show that the number of elemts in $C(\lambda)$ is $n!/z(\lambda)$. Here we define

$$z(\lambda) = \prod_r r^{m_r} \cdot m_r!$$

where m_r = # of times that r occurs in $\lambda = (\lambda_1, \dots, \lambda_k)$

factorial: $i! = i(i-1)(i-2)\dots 1$

Define M^λ to be the v.s. w/ basis the tabloids $\{T\}$ of shape λ, w/ $\lambda \vdash n$. Since S_n acts on the set of tabloids it acts on M^λ.
For each numbering T of λ there is an elemt v_T in M^λ defined by

$$v_T = b_T \cdot \{T\} = \sum_{q \in C(T)} \operatorname{sgn}(q) \{q \cdot T\}$$

Exercise 3: Show that $\sigma \cdot v_T = v_{\sigma \cdot T}$ for all T and all $\sigma \in S_n$.

Define the <u>Specht module</u> S^λ to be the subspace of M^λ spanned by the elemts v_T as T varies over all numberings of λ.
Exercise 3 $\Rightarrow$ S^λ is preserved by S_n, ie. it is a rep of S_n.

Lemma: Let T & T' be numberings of shapes λ and λ', & assume λ does not strictly dominate λ'. If there is a pair of integers in the same row of T' & same column of T then $b_T \cdot \{T'\} = 0$. If there is no such pair, $b_T \cdot \{T'\} = \pm v_T$.

Proof: If there is such a pair, let t be the transposition that switches them. Then

$$\underbrace{b_T \cdot t = -b_T}_{} \quad \text{since } t \in C(T)$$

note that acting by t induces bijection on terms of b_T but now everything has opposite sign

and $t \cdot \{T'\} = \{T'\}$ since $t \in R(T')$. So:

$$b_T \cdot \{T'\} = b_T \cdot (t \cdot \{T'\}) = (b_T \cdot t) \cdot \{T'\} = -b_T \cdot \{T'\} \implies$$

$$b_T \cdot \{T'\} = 0.$$

If no such pair, by Lemma ⊛, $\lambda' = \lambda$ and $\exists\, p' \in R(T')$ and $q \in C(T)$ s.t. $p' \cdot T' = q \cdot T$.

Then $b_T \cdot \{T'\} = b_T \cdot \{p' \cdot T'\} = b_T \cdot \{q \cdot T\} \underset{\text{Exercise 1}}{=} \operatorname{sgn}(q)\, b_T \cdot T$

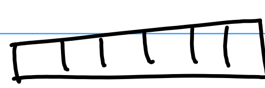
$$= \operatorname{sgn}(q)\, v_T.$$

So $b_T \cdot \{T'\} = \pm v_T$.

Cor: If T and T' are std tableaux w/ $T' > T$ then $b_T \cdot \{T'\} = 0$

Pf: Use Corollary ⊛

Examples of Specht modules.

#1 Let $\lambda = (n)$ = 

ASK! Let $T = \boxed{1 | 2 | 3 | \cdots | \cdots | n}$

$R(T) = S_n$

$C(T) = \{e\}$ so $b_T = \sum_{q \in C(T)} \operatorname{sgn}(q)\, q = e$

There is only one tabloid, $\{T\} = \boxed{1\ 2\ \cdots\ n}$

$V_T = b_T \cdot \{T\} = e \cdot \{T\} = \{T\}$

S_n acts by permuting entries but clearly for any $\sigma \in S_n$, $\sigma \cdot \{T\} = \{T\}$ and so $\sigma \cdot V_T = V_T$.

$\therefore$ trivial rep.

#2 Let $\lambda = (1, 1, \ldots, 1) = \begin{array}{|c|} \hline \ \\ \hline \ \\ \hline \ \\ \hline \vdots \\ \hline \ \\ \hline \end{array}$

Let $T = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$

$R(T) = \{e\}$ and $C(T) = S_n$.

$b_T = \sum_{q \in S_n} \operatorname{sgn}(q)\, q$

Lots of tabloids: $n!$ of them.

$$V_T = b_T \cdot \{T\} = \sum_{q \in S_n} \operatorname{sgn}(q) \{q \cdot T\}$$

For any other tabloid T', $\exists\ \sigma$ s.t. $\{T'\} = \{\sigma . T\}$

so $V_{T'} = V_{\sigma . T} \underset{\text{Exercise 3}}{=} \sigma . V_T$

If we act on V_T by σ we multiply V_T by $\operatorname{sgn}(\sigma)$.

$\therefore$ V_T spans a 1-dim rep of S_n & it is the alternating rep.

Theorem: The S^λ are distinct irreps of S_n.

Proof: No $v_T = 0$ so $S^\lambda \neq 0$.

Let T be any numbering of λ.

Claim 1: $b_T \cdot M^\lambda = \mathbb{C} \cdot v_T \neq 0$.

we have = because for any tabloid T', Lemma :)

$\Rightarrow b_T \cdot \{T'\} = 0$ or $b_T \cdot \{T'\} = \pm v_T$

Claim 2: $b_T \cdot M^\lambda = b_T \cdot S^\lambda$

Why? Clearly $b_T \cdot S^\lambda \subset b_T \cdot M^\lambda$

For any $\{T'\}$ of shape λ, if $b_T \cdot \{T'\} \neq 0$ then $b_T \cdot \{T'\} = \pm v_T$ and

v_T in span of $b_T \cdot v_T = b_T \cdot b_T \cdot \{T\} = \# C(T) \cdot b_T$ by exercise

so each $b_T \cdot \{T'\} \subset b_T \cdot S^\lambda \Rightarrow$
$b_T \cdot M^\lambda \subset b_T \cdot S^\lambda$

$\therefore b_T \cdot S^\lambda = \mathbb{C} \cdot v_T \neq 0$.

lex order

Claim 3: $b_T \cdot S^{\lambda'} = 0$ if $\lambda' > \lambda$

Why? $\lambda' > \lambda \Rightarrow$ for any T' of shape λ', $T' > T \Rightarrow$

by Corollary ✚, $b_T \cdot \{T'\} = 0$.

Thus $b_T \cdot M^{\lambda'} = 0 \Rightarrow b_T \cdot S^{\lambda'} = 0$.

Now assume $S^\lambda = V \oplus W$ not irred.

Then $\mathbb{C} \cdot v_T = b_T \cdot S^\lambda = b_T \cdot (V \oplus W) = b_T \cdot V \oplus b_T \cdot W$.

$\mathbb{C} \cdot v_T$ is one-dim $\Rightarrow$

WLOG $\mathbb{C} \cdot v_T = b_T \cdot V \Rightarrow$

$v_T \in b_T \cdot V \subset V \Rightarrow v_T \in V$.

Note: $S^\lambda = A \cdot v_T$ for any numbering T of λ.
Because each $\sigma \in S_n$ is in A and $\sigma \cdot v_T = v_{\sigma \cdot T}$ by exercise 3 so $A \cdot v_T$ includes all $v_{T'}$ for all numberings T' of λ.

So now $v_T \in V \Rightarrow A \cdot v_T \in V$ (V an A-module) $\Rightarrow S^\lambda \subset V \Longrightarrow S^\lambda = V. \Rightarrow W = 0$.

$\therefore$ S^λ is irreducible!

Note that the S^λ's must be distinct since for a numbering T of λ,

$$b_T \cdot S^\lambda \neq 0 \quad \text{but} \quad b_T \cdot S^{\lambda'} = 0$$

(ie. A acts differently on S^λ and $S^{\lambda'}$). $\blacksquare$

We have found distinct reps for each $\lambda \vdash n$ so we have found all irreps of S_n.

Recall: $f^\lambda = \#$ std tableaux of shape λ.

Theorem: $$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

We will prove this combinatorially next time ...

Prop: The elements v_T, as T varies over the std tableaux on λ, form a basis for S^λ.

Proof: V_T is a linear comb. of $\{T\}$ (w/ coeff 1) and elemts of the form $\{q \cdot T\}$ for $q \in C(T)$ w/ coeffs ± 1. When T is a standard tableau, $q \cdot T < T$ in ordering defined earlier, for $q \neq e$, $q \in C(T)$.

Then the V_T are indep:
Suppose $\sum x_T V_T = 0$ where T varies over std tableau. Look at the maximal T occurring w/ nonzero coeffs in this relation. Then T is a std tableaux, $\{T\}$ occurs w/ coeff 1 in V_T, and $\{T\}$ can't appear w/ nonzero coeff in any other $V_{T'}$.
(all other $T' < T$ and hence $q \cdot T' < T$)

$\therefore \dim S^\lambda \geq f^\lambda$.

But now since we know that
$$n! = \sum_{\lambda \vdash n} (\dim S^\lambda)^2 \geq \sum_{\lambda \vdash n} (f^\lambda)^2 = n!,$$
we must have equality above,
i.e. $\dim S^\lambda = f^\lambda \Rightarrow$
the $\{V_T \mid T \text{ a std tab. of shape } \lambda\}$ form a basis of S^λ.

∎