155 Lecture 20

Alternating sign matrices (ASMs)

(But first, a remark about the last lecture – strong & weak Bruhat order.)

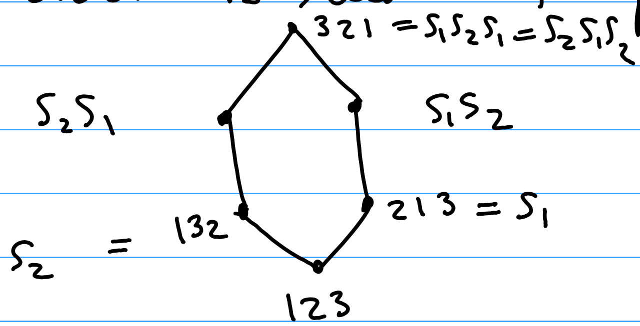
Recall S_n generated by $\{s_i \mid 1 \le i \le n-1\}$ where $s_i = (i \;\; i+1)$.

Weak Bruhat order: we say v precedes w if $\ell(w) = \ell(v) + 1$ and if $\exists$ s_i s.t. $w = v s_i$.

We say $v \le w$ if v and w are connected by a chain of permutations, each of which precedes the next.

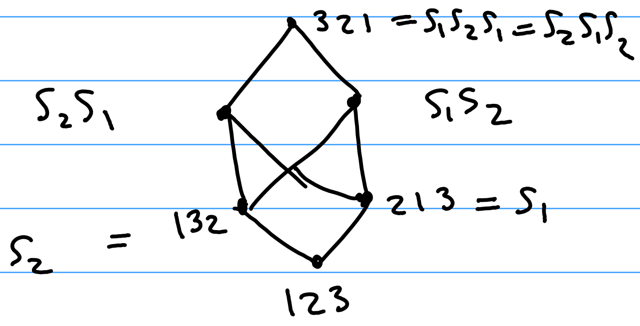
In other words, $v \le w$ if $\exists$ a reduced decomp. of w for which a left factor is a reduced decomp of v.

Weak order is related to permutohedron.



Strong Bruhat order: we say v precedes w if $\ell(w) = \ell(v) + 1$ and if $\exists$ any transp (ij) s.t. $w = v\,(ij)$.
We say $v \le w$ if v and w are connected by a chain of permutations, each of which precedes the next.

One may show that Strong order is characterized by:
Prop: Let $s_{i_1} \dots s_{i_\ell}$ be a reduced decomp of perm w. Then $v \le w$ iff $\exists$ subsequence $(j_1, \dots, j_m)$ of $(i_1, \dots, i_\ell)$ s.t. $s_{j_1} \dots s_{j_m}$ is a red. decomp. of v.



Def: An ASM is a square matrix of 0's, 1's and -1's in which the nonzero entries in each row or column begin & end w/ 1 & alternate in sign.

Ex:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Note: Generalization of permutation matrix

- Discovered by David Robbins & Howard Rumsey in the early 1980's. Came from their analysis of Charles Dodgson's (CS Lewis) <u>condensation</u>

- Desnanot-Jacobi Adjoint Matrix Theorem:
  If M is a matrix, let M^i_j denote M w/ i^{th} row & j^{th} column deleted. Similarly for $M^{ij}_{k\ell}$ (delete 2 rows & 2 columns).

$$(\det M)(\det M^{1n}_{1n}) = (\det M^1_1)(\det M^n_n) - (\det M^1_n)(\det M^n_1)$$

This rule leads inductively to the condensation method of evaluating determinants:

Idea: Start w/ $n \times n$ matrix M, then compute a $(n-1)\times(n-1)$ matrix, then $(n-2)\times(n-2)$ ... eventually get a 1×1 matrix whose sole entry is $\det M$.

Algorithm: For k s.t. $n-1 \geq k \geq 1$, take the k^2 2×2 connected subdeterminants of the $(k+1)\times(k+1)$ matrix & divide them by the corresponding k^2 central entries of the $(k+2)\times(k+2)$ matrix. [In the case $k=n-1$, omit the division step.]

Ex: $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$\downarrow$$

$$\begin{pmatrix} ae-bd & bf-ce \\ dh-eg & ei-fh \end{pmatrix}$$

$$\downarrow$$

$$\frac{(ae-bd)(ei-fh)-(bf-ce)(dh-eg)}{e} =$$

$$\frac{ae^2i - aefh - bdei + bdfh - bdfh + befg + cdeh - ce^2g}{e} =$$

$$aei - afh - bdi + (0)bde^{-1}fh + bfg + cdh - ceg.$$

6 of the terms here correspond to permutation matrices, eg. $aei \Longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(take exponents of the variables occurring in each monomial + place in corresp. position in the matrix)

But one extra term (w/ 0 coefficient) $bde^{-1}fh$. If we do this same procedure get

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$ an ASM

which is not a perm. matrix.

<u>Fact:</u> The terms (including those w/ coeff 0) that one gets by performing condensation are <u>exactly</u> the ASM's.

$n \times n$ ASM's are called ASM's of <u>order n</u>.

Robbins, Mills & Rumsey computed the # of ASM's of order n (by computer) & got the sequence 1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, ...

The growth rate of the sequence & the fact that those #'s factored as products of small primes ($911835460 = 2^2 \cdot 5 \cdot 17^2 \cdot 19^3 \cdot 23$) suggested that there was a formula for the # of ASM's of order n as a ratio of products of factorials.

To find the formula, they divided the set of $n \times n$ ASM's into groups based on the position of 1 in the first row.

Got a triangular array in which k^{th} entry of n^{th} row = # of $n \times n$ ASM's w/ a 1 in row 1, column k.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 3 & & 2 & & & \\
 & & 7 & & 14 & & 14 & & 7 & & \\
 & 42 & & 105 & & 135 & & 105 & & 42 & \\
429 & & 1287 & & 2002 & & 2002 & & 1287 & & 429
\end{array}
$$

$A_{1,1}$, $A_{2,1}$, $A_{2,2}$, $A_{3,1}$, $A_{3,2}$, $A_{3,3}$

$$\frac{A_{n,i}}{A_{n,i+1}}$$

By construction: Sum of entries in row i is the # of ASM's of order i

Note: Entries in southwest diagonal are exactly the sequence of #'s of ASM's. (Ask why!) These entries represent # of $n \times n$ ASM's w/ a 1 in top left corner. This 1 is then the unique 1 in that row & that column & the rest of that matrix (rows 2 to n & columns 2 to n) must form a $n-1 \times n-1$ ASM.

MRR looked at ratios of horizontally adjacent entries.

$$\begin{array}{ccccccccccc} & & & & 2/3 & & 3/2 & & & & \\ & & & 1/2 & & 1 & & 2 & & & \\ & & 2/5 & & 7/9 & & 9/7 & & 5/2 & & \\ & 1/3 & & 9/14 & & 1 & & 14/9 & & 3 & \\ 2/7 & & 11/20 & & 5/6 & & 6/5 & & 20/11 & & 7/2 \end{array}$$

& realized that they could rewrite this as:

$$\begin{array}{ccccccccccc} & & & & & 2/2 & & & & & \\ & & & & 2/3 & & 3/2 & & & & \\ & & & 2/4 & & 5/5 & & 4/2 & & & \\ & & 2/5 & & 7/9 & & 9/7 & & 5/2 & & \\ & 2/6 & & 9/14 & & 16/16 & & 14/9 & & 6/2 & \\ 2/7 & & 11/20 & & 25/30 & & 30/25 & & 20/11 & & 7/2 \end{array}$$

<u>Observation</u>: There is a Pascal-triangle-like relation among the entries above:
If we define $\frac{a}{b} \circledast \frac{c}{d} = \frac{a+c}{b+d}$ then

any entry C above is equal to $\frac{a}{b} \circledast \frac{c}{d}$ where $\frac{a}{b}$ and $\frac{c}{d}$ are the entries just above C.

Also, there is a clear pattern for the diagonals.

Let $A_{n,k}$ = # of $n \times n$ ASM's where the 1 in the first row is in column k.

Then the triangle of quotients we are looking at has entries equal to $\dfrac{A_{n,i}}{A_{n,i+1}}$:

$$
\begin{array}{ccccc}
 & & A_{2,1}/A_{2,2} & & \\
 & A_{3,1}/A_{3,2} & & A_{3,2}/A_{3,3} & \\
A_{4,1}/A_{4,2} & & A_{4,2}/A_{4,3} & & A_{4,3}/A_{4,4} \\
\iddots & & \vdots & & \ddots
\end{array}
$$

Our observations are saying that:

- $$\frac{A_{n+1,i}}{A_{n+1,i+1}} = \frac{A_{n,i-1}}{A_{n,i}} \circledast \frac{A_{n,i}}{A_{n,i+1}} \quad \text{for } 2 \le i \le n-1$$

- $$\frac{A_{n,1}}{A_{n,2}} = \frac{2}{n}$$

- $$\frac{A_{n,n-1}}{A_{n,n}} = \frac{n}{2}$$

Looking at the numerators of

$$
\begin{array}{ccccccccccc}
 & & & & & 2/2 & & & & & \\
 & & & & 2/3 & & 3/2 & & & & \\
 & & & 2/4 & & 5/5 & & 4/2 & & & \\
 & & 2/5 & & 7/9 & & 9/7 & & 5/2 & & \\
 & 2/6 & & 9/14 & & 16/16 & & 14/9 & & 6/2 & \\
2/7 & & 11/20 & & 25/30 & & 30/25 & & 20/11 & & 7/2
\end{array},
$$

one might observe that the array of numerators can be written as

$$
\begin{array}{ccccccccccc}
 & & & & & 1+1 & & & & & \\
 & & & & 1+1 & & 1+2 & & & & \\
 & & & 1+1 & & 2+3 & & 1+3 & & & \\
 & & 1+1 & & 3+4 & & 3+6 & & 1+4 & & \\
 & 1+1 & & 4+5 & & 6+10 & & 4+10 & & 1+5 & \\
1+1 & & 5+6 & & 10+15 & & 10+20 & & 5+15 & & 1+6
\end{array}
$$

So the numerator in row n, position i is: $\binom{n-1}{i-1}+\binom{n}{i-1}$

And the denominators are the same after reflecting over y-axis.

So the denominator in row n, position i is $\binom{n-1}{n-i}+\binom{n}{n-i}$

Since the ratio in row n, position i is conjectured to be $A_{n+1,i}/A_{n+1,i+1}$, this leads to conjecture that

$$\frac{A_{n+1,i}}{A_{n+1,i+1}} = \frac{\binom{n-1}{i-1}+\binom{n}{i-1}}{\binom{n-1}{n-i}+\binom{n}{n-i}} =$$

$$\frac{\dfrac{(n-1)!}{(i-1)!\,(n-i)!} + \dfrac{n!}{(i-1)!\,(n-i+1)!}}{\dfrac{(n-1)!}{(n-i)!\,(i-1)!} + \dfrac{n!}{(n-i)!\,i!}} =$$

$$\frac{(n-1)!}{(i-1)!\,(n-i)!}\left[1+\frac{n}{n-i+1}\right] \div \frac{(n-1)!}{(n-i)!\,(i-1)!}\left[1+\frac{n}{i}\right] =$$

$$\frac{(n-1)!}{(i-1)!\,(n-i)!} \cdot \frac{2n-i+1}{n-i+1} \cdot \frac{(n-i)!\,(i-1)!}{(n-1)!} \cdot \frac{i}{i+n} =$$

$$\frac{\cancel{(n-1)!}}{\cancel{(i-1)!}\,\cancel{(n-i)!}} \cdot \frac{2n-i+1}{n-i+1} \cdot \frac{\cancel{(n-i)!}\,\cancel{(i-1)!}}{\cancel{(n-1)!}} \cdot \frac{i}{i+n} =$$

$$\frac{i\,(2n-i+1)}{(n-i+1)(n+i)}$$

This is the Refined ASM Conjecture:

Conj 1: $$\frac{A_{n+1,i}}{A_{n+1,i+1}} = \frac{i\,(2n-i+1)}{(n-i+1)(n+i)}$$

If Conj. is correct, then the value of each $A_{n,k}$ is uniquely determined by $A_{n,k-1}$ when $k > 1$ and by

$$A_{n,1} = \sum_{k=1}^{n-1} A_{n-1,k}$$

It can be shown that the Conj. above (nontrivial) is equivalent to the following:

Conj 2: For $1 \le k \le n$,

$$A_{n,k} = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}$$

Since $A_{n,1} = A_{n-1}$, Conj 2 $\Rightarrow$ a formula for the # of ASM's of order n.

$$A_{n+1,1} = \binom{n+1+1-2}{1-1} \frac{(2n+2-1-1)!}{(n+1-1)!} \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+1+j)!}$$

$$= \frac{(2n)!}{n!} \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j+1)!}$$

$$= \frac{(2n)!}{n!} \frac{\prod_{j=0}^{n-1} (3j+1)!}{(n+1)!(n+2)! \ldots (2n)!}$$

$$= \frac{\prod_{j=0}^{n-1} (3j+1)!}{n!(n+1)! \ldots (2n-1)!}$$

$$= \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

Therefore,

a consequence of this refined conjecture is the formula

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

This is the ASM Conjecture.
Unproven until 1995 when a team of 88 people checked Zeilberger's 100+ proof of it.

The same year, Greg Kuperberg found a much simpler proof that relied on the Yang-Baxter equation for the 6-vertex model.

Descending Plane Partitions

MRR told their conjecture to Richard Stanley, who told them the same sequence had recently arisen in work of George Andrews on a seemingly unrelated problem in the theory of plane partitions.

Plane partitions were defined by Percy MacMahon in the 1800's, as a generalization of ordinary partitions.

Euler had shown that if we set
$p(n)$ = # of partitions of n and consider the formal power series

$\sum_{n\geq 0} p(n)q^n$, then $\sum_{n\geq 0} p(n)\, q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}$

<u>Proof</u>: $\prod_{k=1}^{\infty} \frac{1}{1-q^k} = \prod_{k=1}^{\infty} (1+q^k+q^{2k}+q^{3k}+\dots)$

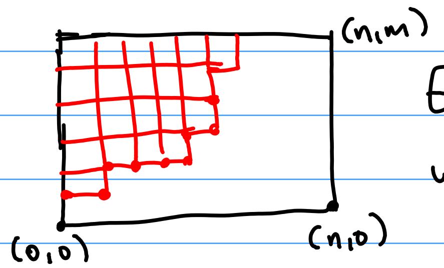
Each term on the RHS comes from choosing a multiple of k for each $k=1,2,\dots$. If for each k we choose a non-negative integer r_k (corresponding to our multiple $r_k k$) then this term is in bijection w/ the partition $(1^{r_1}, 2^{r_2}, 3^{r_3}, \dots)$

The partition where we have chosen r_k parts of length k.

$\therefore$ RHS $= \sum_{n\geq 0} p(n)\, q^n$

We represent partitions as Young diagrams, or as weakly decreasing sequences of numbers.

It is easy to count the # of Young diagrams contained in a $m\times n$ rectangle. (Ask?!)

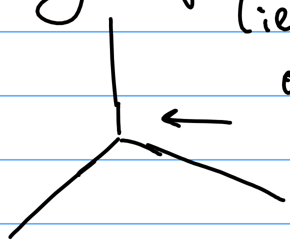


Each such Young diagram $\overset{\text{bijection}}{\longleftrightarrow}$ w/ lattice path which takes

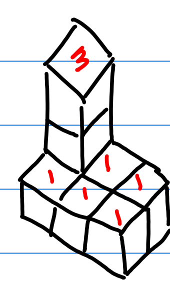
unit steps east + north from $(0,0)$ to (n,m):
$\binom{m+n}{m}$ such paths $\Rightarrow$ $\binom{m+n}{m}$ such Young diagrams.

MacMahon's idea: generalize these 2-dim Young diagrams to 3-dim Young diagrams.

Define a 3-dim Young diagram to be a configuration of cubes inside an octant s.t. each cube is supported on the 3 sides towards the bounding planes of the octant.
(ie. all cubes are pushed as far towards one corner of a room as possible)



Ex:



label stacks of cubes w/ the height

These configurations of cubes $\leftrightarrow$ partitions of a number (the total # of cubes) into parts arranged 2-dimensionally.

Ex above $\leftrightarrow$
$$\begin{matrix} 3 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}$$

So equivalently, we can define a 3-dim Young diagram (or plane partition) to be a 2D array of pos. #'s s.t. both rows + columns weakly decrease + the array of #'s lies in a Young diagram

If T is a plane partition whose entries add to n, we say T is a plane partition of n.

Let $pp(n)$ = # plane partitions of n

Mac Mahon Theorem #1: $\displaystyle\sum_{n\geq 0} pp(n) q^n = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}$

$$= 1 + q + 3q^2 + 6q^3 + 13q^4$$

Let $pp(a\times b\times c)$ = # plane partitions whose 3D diagram fits inside an $a\times b\times c$ box.

Mac Mahon Theorem #2: $\displaystyle pp(a\times b\times c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$

MM #1 + 2 can be proved w/ symm. function + RSK — see Stanley's EC2.

In the 1960's, various mathematicians tried to enumerate different classes of plane partitions whose solid Young diagrams were invariant under some symmetry (eg a reflection).

This led indirectly to notion of descending plane partitions (DPP's).

Def: A DPP of order n is a 2-D array of pos. integers $\leq n$ s.t. the left-hand edges are successively indented, there is a weak decrease across rows & strict decrease down columns, & the # of entries in each row is strictly less than the largest entry in that row.

Example: (order 7)

7 7 6 6 3 1
6 5 4 2
3 3
2

32

There are 7 DPP's of order 3.

ϕ 2 3 31 32 33

33
2

George Andrews found & proved a formula for # of DPP's of order n, which gave him the sequence

$$1, 2, 7, 42, 429, 7436 \ldots$$

His formula was essentially the formula $A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$ ——

Same #'s that seemed to enumerate the ASM's.

Mills, Robbins & Rumsey tried to prove the ASM Conj by finding a bijection between ASM's & DPP's.

However, they were unsuccessful & even now no one has found such a bijection!