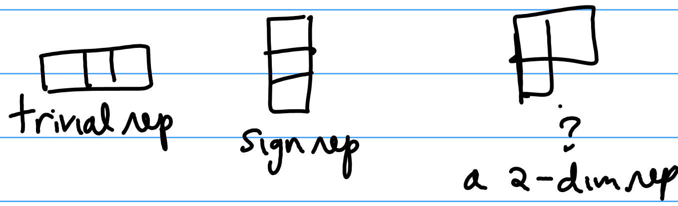
155 Lecture 2

Last time: reviewed def's of representation, irreducible rep, etc.

Stated fact: that irred reps of $S_n \longleftrightarrow$ partitions of n.

Eg for $n=3$, we have



Let's find the other rep of S_3.

Since S_3 is a permutation group, we can let $\{e_1, e_2, e_3\}$ be a basis for $\mathbb{C}^3$ & let G act on $V = \mathbb{C}^3$ $g.e_i = e_{g(i)}$.

In other words, the map $\rho: G \to GL(\mathbb{C}^3)$ is given by:

$$\mathrm{id} \longmapsto \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

ASK

$$(12) \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (23) \mapsto \qquad (13) \mapsto$$

$$(132) \mapsto$$

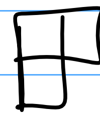
$$(123) \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{etc.}$$

ASK

However, this rep is not irreducible — why? What is the invariant subspace?

The line spanned by $e_1 + e_2 + e_3$ is invariant!

So $\mathbb{C}\langle e_1+e_2+e_3\rangle$ is a subrep of V.
There is a complementary subspace
$W = \{ (z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1+z_2+z_3 = 0\}$
which is invariant under S_3.
This is irreducible & is called the <u>standard representation</u> of S_3.

It has dim 2 and $\longleftrightarrow$ 

If we choose basis for W, can write out the matrices $\rho(\pi)$ for $\pi \in S_3$

eg choose $w_1 = (1,-1,0)$ and $w_2 = (0,1,-1)$.
Then $(12) \cdot w_1 = (-1,1,0) = -w_1$
$(12) \cdot w_2 = (1,0,-1) = w_1 + w_2$ so

$$\rho((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} .$$

Of course $\rho(\mathrm{id}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Similarly $\rho((123)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, etc.

note: we will use these matrices a little later.

Of course if we choose a different basis for W, the homomorphism
$\rho' : S_3 \longrightarrow GL_2(\mathbb{C})$ will be different
ρ' will be related to ρ as follows: there will exist an invertible matrix T s.t.
$\rho'(\pi) = T\rho(\pi)T^{-1}$ for all π.
ie. ρ and ρ' just differ by a change of basis.

Note: For any n, S_n has a standard representation of dim $n-1$:
$W = \{(z_1, .., z_n) \in \mathbb{C}^n \mid z_1 + z_2 + \dots + z_n = 0\}$
S_n acts on W by permuting coordinates.

Now: we will prove that the only 3 irreps of S_3 are the 3 we've seen ... we will use an approach of looking at a large abelian subgp w/in our group — because rep theory of abelian groups is easy! Will come back to this technique later, + use it for all S_n

We have an abelian subgp $U_3 \cong \mathbb{Z}/3\mathbb{Z}$ within S_3 — generated by any 3-cycle τ.
Consider any rep W of S_3 and then think of it as a rep of U_3.
Since U_3 abelian, as a rep of U_3, W decomposes into sum of 1-dimensional irreps.
Each 1-dim rep is spanned by an eigenvector of τ, + eigenvalues must be powers of cube root of unity $\omega = e^{2\pi i/3}$
(because each rep is given by a homomorphism from $\langle 1, \tau, \tau^2 \rangle$ to $\mathbb{C}^*$ and τ has order 3)

So $W = \bigoplus V_i$ where $V_i = \mathbb{C} v_i$ and $\tau v_i = \omega^{\alpha_i} v_i$.

Now: how do the other elements of S_3 act on W in terms of this decomposition?
Let σ be any transposition ((12) or (13) or (23))
Then σ and τ generate S_3 and

(✱) $\sigma \tau \sigma = \tau^2$:

$$\underset{(ab)(abc)(ab) = (acb)}{\sigma \tau \sigma} = \underset{(abc)^2 = (acb)}{\tau^2}$$

If v is an eigenvector for τ w/ eigenvalue ω^i, where does σv go w/in $W = \bigoplus V_i$?

Consider quantity $\tau \sigma v$:

$$\begin{aligned} \tau \sigma v &= \sigma \tau^2 v \quad \text{by (✱)} \\ &= \sigma \omega^{2i} v \\ &= \omega^{2i} \sigma v. \end{aligned}$$

That is, σv is an eigenvector for τ w/ eigenvalue ω^{2i}.

Now we will build all possible irreps of S_3.
Start w/ eigenvector v for τ w/ eigenvalue ω^i.

$\boxed{\text{Case 1}}$ If $\omega^i \neq 1$ then $\omega^{2i} \neq \omega^i$ — so v and σv must be linearly independent, having different eigenvalues for τ.

Claim: $\langle v, \sigma v \rangle$ is invariant under S_3.
Invariant under σ because $\sigma^2 = id$ and invariant under τ because both are

eigenvectors of τ. But σ + τ generate S_3 so invariant under S_3!
So Case 1 $\Rightarrow$ we get 2-dim irrep of S_3.

$\boxed{\text{Case 2}}$ If $\omega^i = 1$ then $\omega^{2i} = 1$ so v and σv are both eigenvectors for τ w/ eigenvalue 1. σv may or may not be indep of v.

Case a If not indep, we have 1-dim rep since σ and τ preserve $\langle v \rangle$.
In that case, $\sigma v = v$ or $-v$:
(σ has order 2 so acts by mult by ± 1)
$\sigma v = v \Rightarrow \tau$ and σ act by mult by 1 $\Rightarrow$ trivial rep
$\sigma v = -v \Rightarrow \tau$ acts by mult by 1 and σ acts by mult by -1 $\rightarrow$ sign rep.

Case b If σv and v are indep then

Claim: $v + \sigma(v)$ spans 1-dim rep $\cong$ to the trivial rep, and $v - \sigma(v)$ spans 1-dim rep $\cong$ to the sign rep.

Pf: $v + \sigma(v)$ is eigenvector for τ w/ eigenvalue 1

Same for σ, because $\sigma(v+\sigma(v)) = \sigma(v) + \sigma^2(v) = \sigma(v) + v$.

$v - \sigma(v)$ is eig for τ w/ eig 1

$v - \sigma(v)$ is eig for σ w/ eig -1 because $\sigma(v - \sigma(v)) = \sigma(v) - v$.

Example suggests that if we have a rep V of G, Knowing the eigenvalues of each $g \in G$ should allow us to describe the rep.

In fact, specifying the set of all eigenvalues of each $g \in G$ is redundant.

If we know all eigenvalues $\{\lambda_i\}$ of a $g \in G$ then we know all eigenvalues of g^k : $\{\lambda_i^k\}$

Key observation! If we know just the sum of the eigenvalues of each $g \in G$, we can recover all eigenvalues of each $g \in G$:
Because if we know the sums $\sum \lambda_i^k$ of the k^{th} powers of the eigenvalues of a given g, we can recover each λ_i.

eg can solve $\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = a \\ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = b \\ \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = c \end{cases}$ for $\lambda_1, \lambda_2, \lambda_3$

So this suggests the following definition:

Def: If V is a rep of G, its character χ_V is the complex-valued function on the group defined by

$$\chi_V(g) = Tr(g|_V),$$ the trace of g on V.

(*) Recall: trace = sum of eigenvalues so χ_V = sum of eig

Note: $\chi_V(hgh^{-1}) = \chi_V(g)$ so χ_V is constant on conjugacy classes of G; such a function is called a class function

Note: $\chi_V(e) = \dim V$.

Why??

$\rho(e)$ is the $n \times n$ identity matrix where $n = \dim V$.

So trace is n.

Prop: Let V and W be reps of G. Then
$\chi_{V \oplus W} = \chi_V + \chi_W$ and $\chi_{V \otimes W} = \chi_V \cdot \chi_W$

Proof: Let $\{\lambda_i\}$ be the eigenvalues of g on V and let $\{\mu_j\}$ be the " " of g on W.
Then the eigenvalues of g on $V \oplus W$ are $\{\lambda_i\} \cup \{\mu_j\}$.
And the eigenvalues of g on $V \otimes W$ are
$\{\lambda_i \mu_j \mid \text{for all } i, \text{ all } j\}$.

Formulas follow since char. is sum of eigenvalues.

Prop If V is the perm rep associated to the action of a group G on a finite set X, show that $\chi_V(g)$ is the number of elemts of X fixed by g.

Proof: Each $\rho(g)$ will be a permutation matrix. We only get a nonzero elemt in the diag when it is a 1, meaning ρ fixes the corresponding element of X

As mentioned, the character is really a function on the conjugacy classes of a group. So we can express the information of characters in a character table.

Put conjugacy classes of G across the top, usually given by a representative g, w/ the # of elemts in each conjugacy class over it.
And list the irreps V of G on the left.
In appropriate box, list the value of χ_V on the corresp. conjugacy class.

Fact: In S_n, conjugacy classes are determined by cycle structure.
eg. in S_5, all perm's w/ this cycle structure are conjugate: $(abc)(de)$

$\therefore$ Conjugacy classes in S_n are specified by giving partitions of n.
For S_4: $(abcd)$ or $(abc)(d)$ or $(ab)(cd)$
or $(ab)(c)(d)$ or $(a)(b)(c)(d)$.

Ex: Char table of S_3:

	1 e	3 (12)	2 (123)
trivial	1	1	1
sign	1	-1	1
standard	2	0	-1

← Ask!!

Keep this table around!

Earlier we looked at std rep & after choosing basis, found that $\rho((12)) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, $\rho((123)) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.
Of course $\rho(\mathrm{id}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\therefore$

Now: State main theorems about characters + see examples.

Let $\mathbb{C}_{class}(G) = \{$class functions on $G\}$

Define Hermitian inner product on $\mathbb{C}_{class}(G)$ by $(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)}\, \beta(g)$.

Theorem: In terms of this inner product, the characters of the irred. reps of G are orthonormal. i.e.

$$(\chi_V, \chi_W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{else} \end{cases}$$

Look at table I drew for S_3...

Cor: The # of irreps of G is $\leq$ # of conjugacy classes. Because dim of space of class function = # of conjugacy classes & the char's of irreps are independent.

Theorem: In fact # of irreps = # of conj. classes

Start char table of S_4 ... ?

Cor: Any representation is determined by its character.

Pf: If $V \cong V_1^{\oplus a_1} \oplus \dots \oplus V_k^{a_k}$ w/ the V_i

distinct irreps, then $\chi_V = \sum a_i \chi_{V_i}$.
No other non-isomorphic rep has this character since the χ_{V_i}'s are lin indep.

Cor: A rep V is irred. iff $(\chi_V, \chi_V) = 1$.
Pf: If $V = V_1^{\oplus a_1} \oplus \dots \oplus V_k^{\oplus a_k}$ then
$(\chi_V, \chi_V) = a_1^2 + \dots + a_k^2$.
So this $= 1$ iff all $a_i = 0$ except some $a_j = 1$.

Cor: The multiplicity a_i of V_i in V is the inner product of χ_V with χ_{V_i}, i.e. $a_i = (\chi_V, \chi_{V_i})$.

Consider regular rep R of G. ← G acting on G. By the previous prop. about permutation representations, we have:

$$\chi_R(g) = \begin{cases} 0 & \text{if } g \neq e \\ |G| & \text{if } g = e \end{cases} \quad = \# \text{ of elemts of } G \text{ that } g \in G \text{ fixes}$$

Write $R = \bigoplus V_i^{\oplus a_i}$ w/ V_i distinct irreps.
Then $$a_i = (\chi_R, \chi_{V_i}) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_R(g)} \chi_{V_i}(g)$$
$$= \frac{1}{|G|} |G| \chi_{V_i}(e) = \chi_{V_i}(e) = \dim V_i$$

∴ Cor: Any irrep V_i of G appears in the regular rep R exactly $\dim V_i$ times. !

dim of regular rep is $|G|$. So...

Cor: $|G| = \dim R = \sum_i (\dim V_i)^2$ where sum is over all irreps of G.

Let f^λ = # standard tableaux of shape λ.

Assuming we know that
irreps V_λ of $S_n \Leftrightarrow$ partition λ of n
and

$\dim V_\lambda = f^\lambda$,

this says that $n! = \sum_{\lambda \vdash n} (f^\lambda)^2$

notation means that λ is a partition of n.

This identity can be proved in combinatorics using remarkable algorithm called RSK.

Example: compute char table of S_4.

5 conj. classes because 5 partitions of 4...
reps are e, (12), (123), (1234), $(12)(34)$

we have trivial, alternating, & standard reps,...
know we have total of 5 irreducibles.

S_4	size 1	6	8	6	3
	e	(12)	(123)	(1234)	$(12)(34)$
trivial U	1	1	1	1	1
alt U'	1	-1	1	-1	1
std V	3	1	0	-1	-1

std rep: S_4 acting on $\{(z_1, z_2, z_3, z_4) \mid \sum z_i = 0\}$

Basis $v_1 = (1,-1,0,0)$ $\quad v_2 = (0,1,-1,0)$, $\quad v_3 = (0,0,1,-1)$

$$\rho(e) = \mathrm{id}, \quad \rho((12)) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho((123)) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho((1234)) = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \qquad \rho((12)(34)) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$(v_2 \mapsto (1,0,0,-1) = v_1 + v_2 + v_3)$$

Not done yet: $|G| = \sum (\dim V_i)^2$ so $\leftarrow$ # term = # conj. class

$$24 = 1 + 1 + 9 + a^2 + b^2 \Rightarrow$$

$$a^2 + b^2 = 13 \Rightarrow \{a, b\} = \{2, 3\}.$$

Tensor alt. rep U' w/ std rep. V to get $U' \otimes V$ w/ character

$$\chi_{U' \otimes V} = \chi_{U'} \cdot \chi_V = (3, -1, 0, 1, -1).$$

Inner prod w/ itself is $\frac{1}{|G|} \sum_{g \in G} (\chi_{U' \otimes V}(g))^2$

$$= \frac{1}{24}\left(1 \cdot 3^2 + 6 \cdot (-1)^2 + 8 \cdot 0^2 + 6 \cdot (1)^2 + 3 \cdot (-1)^2\right) = 1$$

size of conj. classes $\qquad$ $\therefore$ is irred.

S_4	size 1: e	6: (12)	8: (123)	6: (1234)	3: $(12)(34)$
trivial u	1	1	1	1	1
alt u'	1	-1	1	-1	1
std V	3	1	0	-1	-1
$u'\otimes V$	3	-1	0	1	-1
	2	? a	? b	? c	? d

To get last character, use fact that characters are orthonormal

$2+6a+8b+6c+3d=0$ orthog w/ u
$2-6a+8b-6c+3d=0$ orthog w/ u'

$\Rightarrow 4+16b+6d=0 \Rightarrow 2+8b+3d=0$ ⊛

☺ $6+6a-6c-3d=0$ w/ std V
$6-6a+6c-3d=0$ w/ $u'\otimes V$
$\Rightarrow 12-6d=0 \Rightarrow \boxed{d=2}$

⊛ $\Rightarrow \boxed{b=-1}$

From top, $2+6a-8+6c+6=0 \Rightarrow$
$a+c=0 \Rightarrow c=-a.$

From ☺, $6+6a+6a-6=0 \Rightarrow 12a=0 \Rightarrow \boxed{a=0 \Rightarrow c=0.}$

Done.