155 Lecture 19

The Grassmannian, the complete flag variety, Bruhat order...

The Grassmannian & the complete flag variety are geometric objects that are closely related to the combinatorics (partitions, permutations, symm. functions) that we have studied.

The Grassmannian $Gr_{kn}(\mathbb{C})$ (or $Gr_{kn}(\mathbb{R})$) is the set of all k-dim subspaces of $\mathbb{C}^n$, ie. $\{V \subset \mathbb{C}^n \mid \dim V = k\}$.

It is a
- cell complex
- manifold
- projective variety

More concrete way of thinking about it:
Can represent $V \in Gr_{kn}(\mathbb{C})$ as a full rank $k \times n$ matrix;
the k rows span a k-dim subspace of $\mathbb{C}^n$.

But we consider two such matrices A and A' to be equivalent if they span the same subspace, ie. if $\exists\ P \in GL_k(\mathbb{C})$ s.t. $PA = A'$

Why does $PA = A'$ say that A and A' span the same subspace?

Write $P \in GL_k(\mathbb{C})$ as $\begin{pmatrix} p_1 \\ \hline p_2 \\ \hline \vdots \\ \hline p_k \end{pmatrix}$ where p_i's are the row vectors.

Write $A \in Mat(k,n)$ as $\left(c_1 \middle| c_2 \middle| \cdots \middle| c_n\right)$ where the c_i's are the column vectors.

Then PA =

$$\begin{pmatrix} p_1 \\ \hline p_2 \\ \hline \vdots \\ \hline p_k \end{pmatrix} \left(c_1 \middle| c_2 \middle| \cdots \middle| c_n\right) = \begin{array}{c} \swarrow \text{dot product} \\ \begin{pmatrix} p_1 . c_1 & p_1 . c_2 & \cdots & p_1 . c_n \\ p_2 . c_1 & p_2 . c_2 & \cdots & \\ p_3 . c_1 & p_3 . c_2 & \cdots & p_3 . c_n \end{pmatrix} \end{array}$$

On the RHS, note that each row, say $(p_1 . c_1 \quad p_1 . c_2 \ldots \quad p_1 . c_n)$ is just a linear combination of the rows of $\left(c_1 \middle| c_2 \middle| \cdots \middle| c_n\right)$

$\therefore$ the span of the rows of the RHS is contained in the span of the rows of $\left(c_1 \middle| c_2 \middle| \cdots \middle| c_n\right)$

Since P is invertible, the two row spans are the same.

$\therefore$ we can identify $Gr_{kn}(\mathbb{C})$ with

$GL_k(\mathbb{C}) \backslash Mat(k,n)$ ← full rank $k \times n$ matrices

Example: $Gr_{1n}(\mathbb{C})$.

The set of 1-planes (lines) in $\mathbb{C}^n$, ie. projective space.
The set of full rank $1 \times n$ matrices s.t.
$(a_1, \dots, a_n) \sim (b_1, \dots, b_n)$ if $\exists\ \lambda \in GL_1(\mathbb{C}) = \mathbb{C}^*$ s.t.
$\lambda(a_1, \dots, a_n) = (b_1, \dots, b_n)$.

In general the topology of $Gr_{kn}(\mathbb{C})$ is complicated. But we can partition $Gr_{kn}(\mathbb{C})$ into smaller pieces (actually, cells) which are easy to understand.
This will give $Gr_{kn}(\mathbb{C})$ the structure of a cell complex.

Take any matrix A representing an element of $Gr_{kn}(\mathbb{C})$.
Note that performing row operations on A leaves the corresp. element of $Gr_{kn}(\mathbb{C})$ unchanged.

Algorithm: • Look at each row & locate the last nonzero entry. Then permute rows, ordering them by the position of that nonzero entry.

Each $*$ represents an element of $\mathbb{C}$

Get something like:
$$\begin{pmatrix} * & * & * & * & * & * & * \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Within each row, divide every entry by the last nonzero entry

Get something like:
$$\begin{pmatrix} * & * & * & * & * & * & 1 \\ * & * & * & * & 1 & 0 & 0 \\ * & * & * & * & 1 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- If 2 rows r_i & r_j both have the initial 1 in the same position, replace r_j w/ $r_j - r_i$. (& divide again by last nonzero entry)

Get something like:
$$\begin{pmatrix} * & * & * & * & * & * & 1 \\ * & * & * & * & 1 & 0 & 0 \\ * & * & * & 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- For i from 1 to k, take row r_i & subtract from it appropriate multiples of rows r_j $(i < j \le k)$ in order to put $k-j$ 0's into the new row r_i.

Get something like:
$$\begin{pmatrix} * & 0 & * & 0 & 0 & * & 1 \\ * & 0 & * & 0 & 1 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Deleting the identity submatrix from above we get an array of complex numbers:

$$\begin{matrix} * & * & * \\ * & * & \\ * & * & \\ * & & \end{matrix}$$

Their shape represents a <u>partition</u> (here, $(3,2,2,1)$).

<u>Observation</u>: This algorithm associate to any $A \in Gr_{k,n}(\mathbb{C})$ a unique matrix of the form

$$\begin{pmatrix} * & 0 & * & 0 & 0 & * & 1 \\ * & 0 & * & 0 & 1 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $*$'s represent elemnts of $\mathbb{C}$.

The array of $*$'s forms a partition λ whose Young diagram $\subseteq$ $k \times (n-k)$ rectangle.

All such 0-1-$*$ matrices arise in this way for all Young diagrams $\lambda \subseteq k \times (n-k)$.

<u>Def</u>: If we fix λ — equivalently, fix the form of the matrix as in

$$\begin{pmatrix} * & 0 & * & 0 & 0 & * & 1 \\ * & 0 & * & 0 & 1 & 0 & 0 \\ * & 0 & * & 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The set of all elemts of $Gr_{k,n}(\mathbb{C})$ of this form is called the Schubert cell Ω_λ

Topologically this is just $\mathbb{C}^d$ where $d = |\lambda|$ (# of boxes of λ).

So $$Gr_{k,n}(\mathbb{C}) = \bigsqcup^{*}_{\lambda \subseteq k \times (n-k)} \Omega_\lambda \qquad \text{(disjoint union)}$$

A natural question is: when is one cell $\Omega_\lambda \subset \overline{\Omega_\mu}$?

Answer: $\Omega_\lambda \subset \overline{\Omega_\mu}$ iff $\lambda \subseteq \mu$. (containment of Young diagrams)

Fact (beyond scope of course): The cohomology ring $H^*(Gr_{k,n}(\mathbb{C}))$ is a polynomial ring which is generated by the Schur functions s_λ for all $\lambda \subseteq k \times (n-k)$

Now: Flag varieties.

Fix n and consider a subset $\{i_1 < i_2 < \ldots < i_r\} \subset \{1, \ldots, n\}$.

We define the partial flag variety $Fl(i_1, \ldots, i_r)$ to be the set of flags in $\mathbb{C}^n$

$$\{ V_1 \subset V_2 \subset \ldots \subset V_r \subset \mathbb{C}^n \mid \dim V_j = i_j \}.$$

each V_i a vector space.

Each chain of subspaces $V_1 < \dots < V_r$ is called a <u>flag</u>.

Note that if $\{i_1 < \dots < i_r\} = \{k\}$, the corresponding flag variety is just the Grassmannian $Gr_{k,n}(\mathbb{C})$.

When $\{i_1 < \dots < i_r\} = \{1, \dots, n\}$, this is called the <u>complete flag variety</u> denoted Fl_n.

Just as we did for the Grassmannian, we can represent a flag variety by a matrix. Eg if A is a full rank $n \times n$ matrix, w/ rows

$$\begin{pmatrix} a_1 \\ \hline a_2 \\ \hline \vdots \\ \hline a_n \end{pmatrix},$$

then this corresponds to the complete flag $(\mathrm{span}(a_1), \mathrm{span}(a_1, a_2), \dots, \mathrm{span}(a_1, a_2, \dots, a_n))$.

As w/ the Grassmannian, we can perform certain operations on the matrix which preserve the corresp element of Fl_n.

[Ask what operations are legit!]

We can:
- multiply a row by a scalar
- If $i<j$, we can replace r_j with $r_j - \lambda r_i$.
  But we can't do this if $i > j$ (why?)
  And we can't permute rows (why?)

So there is again an algorithm by which we can take a matrix $\begin{pmatrix} a_1 \\ \hline \vdots \\ \hline a_n \end{pmatrix}$ representing an element of Fl_n & put it in a canonical form.

<u>Algorithm:</u>

- Divide each row by its last nonzero entry, getting something like

Again, $*$'s represent entries in $\mathbb{C}$ $\longrightarrow$

$$\begin{pmatrix} * & * & * & * & 1 & 0 \\ * & 1 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 1 \\ * & * & * & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 \end{pmatrix}$$

- Take row r_j & subtract off appropriate multiples of r_i (for all $i<j$) to put 0's in as many entries as possible of r_j.

Get something like

$$\begin{pmatrix} * & * & * & * & 1 & 0 \\ * & 1 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 0 & 1 \\ * & 0 & * & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Notice that the positions of the 1's defines a permutation π. Here, it is $(5,2,6,4,1,3)$

Note that the $*$'s which remain exactly correspond to entries of the matrix w/ which have a 1 to the right & a 1 below. Letting i & j be the rows of those 1's ($i<j$), these correspond to when $\pi(i) > \pi(j)$.

<u>Def</u>: An <u>inversion</u> of a permutation π is a pair $i<j$ s.t. $\pi(i) > \pi(j)$.

<u>Observation</u>: Each elemt of Fl_n can be uniquely represented in the form

$$\begin{pmatrix} * & * & * & * & 1 & 0 \\ * & 1 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 0 & 1 \\ * & 0 & * & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

for some $\pi \in S_n$ & where each $*$ is replaced w/ an elemt of $\mathbb{C}$.

And all matrices of this form represent elemts of Fl_n.

If we fix $\pi \in S_n$, the set of all elemts of Fl_n represented by

$$\begin{pmatrix} * & * & * & * & 1 & 0 \\ * & 1 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & 0 & 1 \\ * & 0 & * & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

where position of 1's gives π

is called the <u>Schubert cell</u> Ω_π.
Topologically, $\Omega_\pi \cong \mathbb{C}^{\# \mathrm{inv}(\pi)}$

Recall S_n is generated by $\langle s_i \mid 1 \le i \le n-1 \rangle$
where $s_i = (i \;\; i+1)$

<u>Def</u>: The <u>length</u> $\ell(\pi)$ of $\pi \in S_n$ is the minimal number r for which we can write $\pi = s_{i_1} s_{i_2} \dots s_{i_r}$.
Any such minimal factorization of π is called a <u>reduced expression</u> of π.

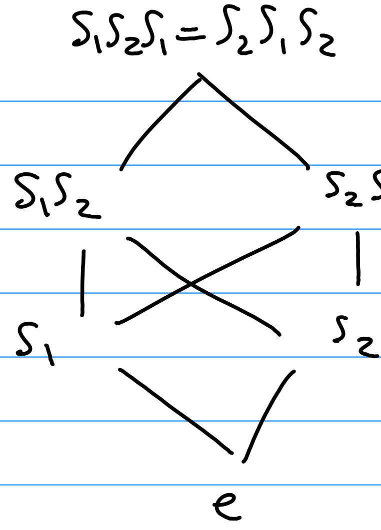
In your homework, you proved (hopefully) that $\ell(\pi) = \#$ inversions of π.

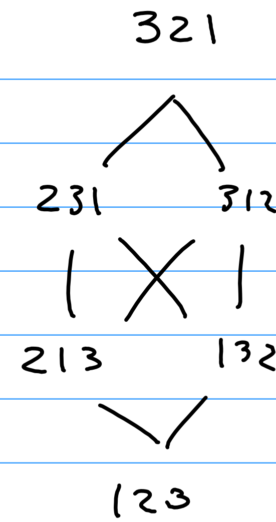
There is more structure on S_n that is related to $\ell(\pi)$: S_n has the structure of a partially ordered set (poset), under the <u>Bruhat order</u>.

<u>Def</u>: Let $s_{i_1} \dots s_{i_\ell}$ be a reduced decomp. of a permutation w. Then we say $v \le w$ iff $\exists$ a subsequence $(j_1, \dots, j_m)$ of $(i_1, \dots, i_\ell)$ s.t. $s_{j_1} \dots s_{j_m}$ is a <u>reduced decomp.</u> of v

This is the Bruhat order. (also known as strong order or strong Bruhat order)

For S_3:





This depiction of the poset is called the <u>Hasse diagram</u>.
We put edges between elements of the poset that are <u>cover relations</u>, i.e. $u < v$ where $\nexists$ w s.t. $u < w < v$.

In this example, there is a <u>rank function</u> on the poset. i.e. Since all maximal chains have the same length, we can define rank(w) to be the # r of elements in a maximal chain $e < w^1 < w^2 < \ldots < w^r = w$.

And for the Bruhat order, rank$(w) = \ell(w)$.

Question: When do we have $\Omega_v \subset \overline{\Omega_w}$? (Schubert cells)

<u>Answer:</u> Exactly when $v \leq w$ in the Bruhat order.