155 Lecture 18

Recall:

Schur/Weyl module construction of some reps of $GL(V)$:

Let V be a fin. dim complex vector space. $GL(V)$ acts on $V^{\otimes d}$ with the diagonal action, ie. $g(v_1 \otimes \dots \otimes v_d) = gv_1 \otimes \dots \otimes gv_d$.

S_d acts on $V^{\otimes d}$ by permuting the factors, ie.

$$(v_1 \otimes \dots \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(d)}.$$

Recall if $\lambda \vdash d$, we can take a filling T of λ, say $\rightarrow$ and define $Row(\lambda)$ & $Col(\lambda)$.

1	2	3	4
5	6	7	
8	9		

Then define 2 elemts in the group alg $\mathbb{C}[S_d]$:

$$a_\lambda = \sum_{w \in Row(\lambda)} w \quad \text{and} \quad b_\lambda = \sum_{w \in Col(\lambda)} sgn(w) \cdot w$$

$c_\lambda = a_\lambda b_\lambda \in \mathbb{C}[S_d]$ Young symmetrizer.

Let $S_\lambda V$ denote the image of c_λ on $V^{\otimes d}$, ie $(V^{\otimes d})c_\lambda$

We saw: This is a representation of $GL(V)$.
(Weyl module)

We saw last time that if $\lambda = (d)$ then

$$S_\lambda V = \mathrm{Sym}^d V \quad \text{and}$$

$$\chi_{S_\lambda V}(g) = h_d(x_1, \dots, x_k) = S_{(d)}(x_1, \dots, x_k)$$

where the x_i's are the eigenvalues of $g \in GL(V)$.

And if $\lambda = (\underbrace{1, 1, \dots, 1}_{d \text{ times}})$ then

$$S_\lambda V = \mathrm{Alt}^d V \quad \text{and}$$

$$\chi_{S_\lambda V}(g) = e_d(x_1, \dots, x_k) = S_{(1,1,\dots,1)}(x_1, \dots, x_k)$$

where the x_i's are the eigenvalues of $g \in GL(V)$.

Theorem: (1) Let $k = \dim V$. Then $S_\lambda V$ is zero if $\lambda_{k+1} \neq 0$. If $\lambda = (\lambda_1 \geq \dots \geq \lambda_k \geq 0)$ then

$$\dim S_\lambda V = S_\lambda(1, \dots, 1) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

(2) $V^{\otimes d} \cong \bigoplus_\lambda (S_\lambda V)^{\oplus f^\lambda}$ where $f^\lambda = \dim$ of the irrep V_λ of S_d

(3) For $g \in GL(V)$, the trace of g on $S_\lambda V$ = value of the Schur poly on the eigenvalues $x_1, \dots, x_k$ of g on V:

$$\chi_{S_\lambda V}(g) = S_\lambda(x_1, \dots, x_k).$$

(4) Each $S_\lambda V$ is an irrep of $GL(V)$.

(we will prove #3; for rest of proof, see Fulton + Harris)

As a parallel story for S_n,
Last time I started trying to convince you that

Claim: Tabloid module $M_\lambda \cong \mathbb{C}[S_d]\, a_\lambda$ and the irrep $V_\lambda \cong \mathbb{C}[S_d]\, c_\lambda$ $\Big\}$ rep's of S_n

[Recall V_λ spanned by $b_T\{T\}$]

Idea: After fixing $\lambda \vdash d$,

Choose the standard filling $\rightarrow$ $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \cline{1-2}\end{array}$.

$$\mathrm{Row}(\lambda) = S_{\{1,2,3\}} \times S_{\{4,5\}}$$

Note: we can represent a permutation w/a tableau — ie. if $\lambda = (3,2)$,

$$u = (4,2,1,5,3) \quad \longleftrightarrow \quad \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 5 & 3 & \\ \cline{1-2}\end{array}$$

The effect of multiplying u by any element in $\mathrm{Row}(\lambda)$ is to permute elemts w/in each row.

So $u \cdot a_\lambda = \underbrace{\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 5 & 3 \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 2 & 4 & 1 \\ \hline 5 & 3 \\ \cline{1-2}\end{array} + \cdots}_{12 \text{ terms}} + \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$

which we can identify w/ the <u>tabloid</u> $\begin{array}{ccc}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$

So $\mathbb{C}[S_d]\, a_\lambda$ = linear combinations of tabloids.

When we multiply $u.a_\lambda$ by b_λ getting $u.a_\lambda b_\lambda$, this element can be identified w/ $b_T \cdot \{T\}$.

This gives a way of constructing a map

$$\varphi: V_\lambda \longrightarrow \mathbb{C}[S_d]\, a_\lambda b_\lambda \quad \text{sending}$$
$$b_T \{T\} \longrightarrow u.\, a_\lambda b_\lambda$$

where u is the perm. corresp. to std tableau T.

[Need to check isomorphism]

<u>Ex:</u> if $\lambda = (d)$ then

$$V_{(d)} = \mathbb{C}[S_d] \cdot \sum_{g \in S_d} e_g = \mathbb{C} \cdot \sum_{g \in S_d} e_g .$$

So $V_{(d)}$ is a 1-dim vector space.

Gets identified w/ the tabloid $\begin{array}{ccccc}\hline 1 & 2 & 3 & \cdots & d \\ \hline\end{array}$

This is the trivial representation.

Part of the proof of the theorem about the rep's $S_\lambda V$ of $GL(V)$:

By def., $S_\lambda V = V^{\otimes d} c_\lambda$ (rep of $GL(V)$),

Also, $V_\lambda = \mathbb{C}[S_d] c_\lambda$ (rep of S_n)

$M_\lambda = \mathbb{C}[S_d] a_\lambda$ (rep of S_n).

By some general nonsense, one shows that if $c \in \mathbb{C}[S_d]$ then

$$V^{\otimes d} c \cong V^{\otimes d} \otimes_{\mathbb{C}[S_d]} \mathbb{C}[S_d] c$$

is an isomorphism of $GL(V)$-modules.

[To construct iso, $(v_1 \otimes \ldots \otimes v_d) c \longmapsto (v_1 \otimes \ldots \otimes v_d) \otimes_{\mathbb{C}[S_d]} c$]

($GL(V)$ acts on RHS by just acting on the $V^{\otimes d}$ part).

Putting $c = c_\lambda$ we get

$$S_\lambda V \cong V^{\otimes d} \otimes_{\mathbb{C}[S_d]} V_\lambda \leftarrow S_d \text{ irrep}$$

Putting $c = a_\lambda$ we get

$$V^{\otimes d} a_\lambda \cong V^{\otimes d} \otimes_{\mathbb{C}[S_d]} M_\lambda$$

(explain afterwards)

Easy to see that $V^{\otimes d} a_\lambda = \mathrm{Sym}^{\lambda_1} V \otimes \ldots \otimes \mathrm{Sym}^{\lambda_k} V$.

So $\mathrm{Sym}^{\lambda_1} V \otimes \ldots \otimes \mathrm{Sym}^{\lambda_k} V \cong V^{\otimes d} \otimes_{\mathbb{C}[S_d]} M_\lambda$.

# of copies of V_μ

Recall Young's Rule: $M_\lambda \cong \bigoplus_\mu K_{\mu\lambda} V_\mu$.

$$\therefore \mathrm{Sym}^{\lambda_1} V \otimes \dots \otimes \mathrm{Sym}^{\lambda_k} V \cong V^{\otimes d} \otimes_{\mathbb{C}[S_\lambda]} \left(\bigoplus_\mu K_{\mu\lambda} V_\mu \right)$$

$$\cong \bigoplus_\mu \left[K_{\mu\lambda} \underbrace{V^{\otimes d} \otimes_{\mathbb{C}[S_\lambda]} V_\mu} \right]$$

$$\cong \bigoplus_\mu K_{\mu\lambda} (S_\mu V)$$

The trace of $g \in GL(V)$ acting on the LHS is

$h_{\lambda_1}(x_1, \dots, x_k) \dots h_{\lambda_k}(x_1, \dots, x_k) = h_\lambda(x_1, \dots, x_k)$,
where $x_1, \dots, x_k$ are eigenvalues of g.

So $h_\lambda(x_1, \dots, x_k) = \sum_\mu K_{\mu\lambda}$ (Trace of g on $S_\mu V$)

But we know $h_\lambda(x_1, \dots, x_k) = \sum_\mu K_{\mu\lambda} s_\mu(x_1, \dots, x_k)$

Schur functions

& this relation is invertible.

$\therefore$ Trace of g on $S_\mu V = s_\mu(x_1, \dots, x_k)$.

Why is $V^{\otimes d} a_\lambda = \mathrm{Sym}^{\lambda_1} V \otimes \dots \otimes \mathrm{Sym}^{\lambda_k} V$?

$$a_\lambda = \sum_{w \in S_{\lambda_1} \times \dots \times S_{\lambda_k}} w. \qquad \lambda \vdash d$$

$$(\underbrace{v_{i_1} \otimes \dots \otimes v_{i_{\lambda_1}}}_{\text{Row 1}} \otimes \underbrace{v_{i_{\lambda_1+1}} \otimes \dots \otimes v_{i_{\lambda_1+\lambda_2}}}_{\text{Row 2}} \otimes \dots \otimes v_{i_d})\, a_\lambda =$$

$\dots$

sum of all $(v_{j_1} \otimes \dots \otimes v_{j_d})$ obtained from this by permuting indices w/in rows only.

This element can be identified with

$$(v_{i_1} \dots v_{i_{\lambda_1}}) \otimes (v_{i_{\lambda_1+1}} \dots v_{i_{\lambda_1+\lambda_2}}) \otimes \dots \otimes (v_{i_{\lambda_1+\dots+\lambda_{k-1}+1}} \dots v_{i_d}) \in$$

$$\mathrm{Sym}^{\lambda_1} V \otimes \mathrm{Sym}^{\lambda_2} V \otimes \dots \otimes \mathrm{Sym}^{\lambda_k} V.$$

Now that we have linked these reps of $GL(V)$ to symmetric functions, we can deduce some corollaries.

Note: The character of a tensor product of 2 reps is the product of the char's of the factors.

$\therefore$ $\mathrm{Char}(S_\lambda V \otimes S_\mu V) = \mathrm{Char}(S_\lambda V)\,\mathrm{Char}(S_\mu V)$

$$= s_\lambda(x_1, \dots, x_k)\, s_\mu(x_1, \dots, x_k)$$

Recall: Littlewood Richardson #'s are the coeffs $c^\nu_{\lambda\mu}$ such that

$$s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$$

Since $s_\lambda s_\mu$ is the char of $S_\lambda V \otimes S_\mu V$, the LR #'s express how

$S_\lambda V \otimes S_\mu V$ decompose into irreps $S_\nu V$

of $GL(V)$.

Now Another way to construct these rep's of $GL(V)$ — as a solution of a universal problem

We need notion of an exchange.
Choose 2 columns of a Young diagram λ & choose i boxes from each column. (any i) For any filling T of λ, the corresponding exchange is the filling S obtained from T by interchanging the entries in the 2 chosen sets of

boxes, maintaining the vert. order in each. The other entries are unchanged.

Eg. if $\lambda = (4,3,3,2)$, & the chosen entries are the top 2 in the 3rd column and the 2nd & 4th box in the second column, the exchange takes

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 5 & 2 & 1 \\ \hline 1 & 3 & 4 \\ \cline{1-3} 2 & 4 & 5 \\ \cline{1-3} 3 & 5 \\ \cline{1-2} \end{array} \longmapsto \begin{array}{|c|c|c|c|} \hline 1 & 5 & 3 & 1 \\ \hline 1 & 2 & 5 \\ \cline{1-3} 2 & 4 & 5 \\ \cline{1-3} 3 & 4 \\ \cline{1-2} \end{array}$$

Let E be a vect. space
Write $E^{\times\lambda}$ for the Cartesian product of $n = |\lambda|$ copies of E, where each copy is labeled by a box of the Young diagram of λ.

So an element v of $E^{\times\lambda}$ is given by specifying an elemt of E for each box of λ.

Consider maps $\varphi : E^{\times\lambda} \longrightarrow F$ (vect. space) s.t.

1. $\mathcal{E}$ is multilinear
2. $\mathcal{E}$ is alternating in the entries of any column of λ, i.e. $\mathcal{E}$ vanishes if 2 entries in the same column are equal
3. For any v in $E^{\times\lambda}$, $\mathcal{E}(v) = \sum \mathcal{E}(w)$, where the sum is over all w obtained from v by an exchange between 2 given columns, w/ a ~~given~~ right chosen subset of boxes in the chosen column.

Note: The 2 columns + the set of boxes in the right column are fixed. If the # of boxes chosen is k + the left chosen column has length c, there are $\binom{c}{k}$ such w for a given v.

Eg For $\lambda = (2,2,2)$, + choosing the top box in the 2^{nd} column, we have

$$\mathcal{E}\left(\begin{array}{|c|c|} \hline x & u \\ \hline y & v \\ \hline z & w \\ \hline \end{array}\right) = \mathcal{E}\left(\begin{array}{|c|c|} \hline u & x \\ \hline y & v \\ \hline z & w \\ \hline \end{array}\right) + \mathcal{E}\left(\begin{array}{|c|c|} \hline x & y \\ \hline u & v \\ \hline z & w \\ \hline \end{array}\right) + \mathcal{E}\left(\begin{array}{|c|c|} \hline x & z \\ \hline y & v \\ \hline u & w \\ \hline \end{array}\right)$$

We define the <u>Schur module</u> E^λ to be the universal target module for such maps φ.

i.e. we have a map $E^{\times\lambda} \to E^\lambda$ that we denote $v \mapsto v^\lambda$ satisfying (1) to (3) & s.t. for any map $\varphi : E^{\times\lambda} \to F$ satisfying (1) to (3) $\exists!$ homomorphism $\tilde{\varphi} : E^\lambda \to F$ s.t. $\varphi(v) = \tilde{\varphi}(v^\lambda)$ $\forall\, v \in E^{\times\lambda}$

<u>Ex:</u> $\lambda = (n)$ get $\mathrm{Sym}^n(E)$
$\lambda = (1, \dots, 1)$ get $\mathrm{Alt}^n(E)$

To construct E^λ from this description, note that the universal module w/ properties ① and ② is
$$\Lambda^{\mu_1}(E) \otimes \dots \otimes \Lambda^{\mu_\ell}(E)$$
where μ is the conjugate partition to λ.

(Property ① alone gives $E^{\otimes\lambda}$ and property ② takes a certain quotient)

The map from $E^{\times\lambda}$ to $\Lambda^{\mu_1}(E) \otimes \dots \otimes \Lambda^{\mu_\ell}(E)$ is the obvious one.

eg. $\begin{array}{|c|c|}\hline x & u \\ \hline y & v \\ \hline z & w \\ \hline\end{array} \mapsto (x \wedge y \wedge z) \otimes (u \wedge v \wedge w)$ in $\Lambda^3(E) \otimes \Lambda^3(E)$

Then $E^\lambda = \left(\Lambda^{\mu_1} E \otimes \cdots \otimes \Lambda^{\mu_\ell} E\right) / Q^\lambda(E)$

where $Q^\lambda(E)$ is the submodule generated by all elemts of the form

$\wedge \underline{v} - \sum \wedge \underline{w}$, the sum over all $\underline{w}$ obtained from $\underline{v}$ by the exchange procedure, for some choice of columns + boxes.

Given this description, it is <u>plausible</u> (but requires proof) that if $\dim V = r$, a basis for E^λ can be constructed which is in bijection w/ semistd tableaux of shape λ filled w/ numbers $\leq r$.