155 Lecture 17

Review of Multilinear algebra: tensor, exterior, & symmetric powers.

Def: The <u>tensor product</u> of 2 vector spaces V & W over a field is a vect. space $V \otimes W$ equipped w/ a bilinear map

$$V \times W \to V \otimes W$$
$$v \times w \longmapsto v \otimes w$$

which is <u>universal</u>:
for any bilinear map $\beta : V \times W \to U$ to a vector space U, there is a unique linear map from $V \otimes W$ to U that takes $v \otimes w$ to $\beta(v, w)$.
This determines the tensor product up to canon. isomorphism.

Explicit construction: if $\{e_i\}$ and $\{f_j\}$ are bases for V and W, the elemts $\{e_i \otimes f_j\}$ form a basis for $V \otimes W$.
ie. $V \otimes W$ consists of all linear combinations of elemts of the form $e_i \otimes f_j$.

Similarly we have the tensor product $V_1 \otimes \dots \otimes V_n$ of n vector spaces.

<u>Properties</u>:
comm: $V \otimes W \cong W \otimes V$ via $v \otimes w \mapsto w \otimes v$

distrib: $(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$

assoc: $(U \otimes V) \otimes W \cong U \otimes (V \otimes W) \cong U \otimes V \otimes W$ via

$$(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \mapsto u \otimes v \otimes w$$

If A is an algebra over the ground field & V is a right A-module & W is a left A-module then we have a tensor product

$V \otimes_A W$:= quotient of $V \otimes W$ by the subspace generated by all $(v \cdot a) \otimes w - v \otimes (a \cdot w)$ $\forall$ $v \in V$, $w \in W$, $a \in A$.

So in $V \otimes_A W$, we have

$$va \otimes_A w = v \otimes_A aw$$

(we can move element of A across the $\otimes$)

Def: A multilinear map β is alternating if $\beta(v_1, \dots, v_n) = 0$ whenever two of the vectors are equal.

Note: B alternating $\Rightarrow B(v_1, \dots, v_n)$ changes sign whenever two of the vectors are switched.
Because: $0 = B(v_1+v_2, v_1+v_2) = B(v_1, v_2) + B(v_2, v_1)$ ✓
$\therefore$ $B(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \operatorname{sgn}(\sigma)\, B(v_1, \dots, v_n) \quad \forall\, \sigma \in S_n$.

The exterior powers $\Lambda^n V$ of a vect. space V, sometimes denoted $\mathrm{Alt}^n V$, come w/ an alternating multilinear map

$$V \times \dots \times V \longrightarrow \Lambda^n V, \quad v_1 \times \dots \times v_n \mapsto v_1 \wedge \dots \wedge v_n$$

that is universal: for $B : V \times \dots \times V \to U$ an alternating multilinear map, $\exists!$ linear map from $\Lambda^n V$ to U which takes $v_1 \wedge \dots \wedge v_n$ to $B(v_1, \dots, v_n)$.

Explicit construction: quotient of $V^{\otimes n}$ by subspace generated by all $v_1 \otimes \dots \otimes v_n$ w/ two of the vectors equal.

If $\{e_i\}$ is a basis for V, then $\{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_n} : i_1 < i_2 < \dots < i_n\}$ is a basis for $\Lambda^n V$.
Define $\Lambda^0 V$ to be the ground field.

Def: A multilinear map $B : V \times \dots \times V \to U$ is symmetric if $B(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = B(v_1, \dots, v_n) \ \forall \sigma \in S_n$.

The symmetric powers $\mathrm{Sym}^n V$ ($S^n V$) comes w/ a univ. symm. multilinear map
$V \times \dots \times V \to \mathrm{Sym}^n V$, $\quad v_1 \times \dots \times v_n \mapsto v_1 \cdot \ldots \cdot v_n$.

This can be constructed as the quotient of $V^{\otimes n}$ by the subspace generated by all
$v_1 \otimes \dots \otimes v_n - v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}$

If $\{e_i\}$ is a basis for V, then
$\{e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_n} : i_1 \le i_2 \le \dots \le i_n\}$ is a basis for $\mathrm{Sym}^n V$.

So $\mathrm{Sym}^n V$ can be regarded as the space of homog. poly's of degree n in the variables e_i.

Define $\mathrm{Sym}^0 V$ to be ground field.

We can also identify $\Lambda^n V$ and $\mathrm{Sym}^n V$ as subspaces of $V^{\otimes n}$, assuming that char $F = 0$.

$i : \Lambda^n V \to V^{\otimes n}$ is defined by

$$i(v_1 \wedge \dots \wedge v_n) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\, v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}$$

$\mathrm{im}\ i \cong \Lambda^n V$.

Similarly we define $i : \mathrm{Sym}^n V \to V^{\otimes n}$ by
$$i(v_1 \cdot \ldots \cdot v_n) := \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}.$$

Now, let's define some representations of $GL(V)$.

Let V be a fin. dim complex vector space. $GL(V)$ acts on $V^{\otimes d}$ with the diagonal action, i.e. $g(v_1 \otimes \dots \otimes v_d) = gv_1 \otimes \dots \otimes gv_d$.

We saw that S_d acts on $V^{\otimes d}$ by permuting the factors, i.e.

$$(v_1 \otimes \dots \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(d)}.$$

Recall if $\lambda \vdash d$, we can take a filling T of λ, say $\longrightarrow$ $\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 & 9 \\ \cline{1-2}\end{array}$ and define $\text{Row}(\lambda)$ & $\text{Col}(\lambda)$.

Then define 2 elemts in the group alg $\mathbb{C}[S_d]$:

$$a_\lambda = \sum_{w \in \text{Row}(\lambda)} w \quad \text{and} \quad b_\lambda = \sum_{w \in \text{Col}(\lambda)} \text{sgn}(w) \cdot w$$

$c_\lambda = a_\lambda b_\lambda \in \mathbb{C}[S_d]$ Young symmetrizer.

Note that S_d acts on $V^{\otimes d}$ by permuting the factors, i.e.

$$(v_1 \otimes \dots \otimes v_d) \cdot \sigma = v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(d)}.$$

Let $S_\lambda V$ denote the image of c_λ on $V^{\otimes d}$, ie $(V^{\otimes d})c_\lambda$

<u>CLAIM</u> This is a representation of $GL(V)$.

<u>Why?</u> Need: for $g \in GL(V)$, $g.S_\lambda V \subset S_\lambda V$. Since actions of $GL(V)$ + S_d on $V^{\otimes d}$ <u>commute</u>, this follows... $g.(V^{\otimes d})c_\lambda \subset (V^{\otimes d})c_\lambda$.

The functor $V \rightsquigarrow S_\lambda V$ is called the <u>Schur functor</u> or <u>Weyl module</u> corresponding to λ.

<u>Ex:</u> If $\lambda = (2)$ then $S_{(2)}V = (\mathrm{id} + (12))V \otimes V$

Elements of $S_{(2)}V$ have the form

$(\mathrm{id} + (12))(v_i \otimes v_j) = v_i \otimes v_j + v_j \otimes v_i$,

we allow $i = j$

which we identify w/ the elemt $v_i \cdot v_j$ of $\mathrm{Sym}^2 V$.

More generally, if $\lambda = (d)$ then $S_\lambda V = \sum_{w \in S_d} w . V^{\otimes d}$

Elemts have the form $\left(\sum_{w \in S_d} w\right)(v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_d})$

which we identify w/ the element $v_{i_1} v_{i_2} \dots v_{i_d}$ of $\mathrm{Sym}^d V$.

Let $x_1, \dots, x_k$ be the eigenvalues of $g \in GL(V)$.
The character χ_{S_λ} of the action of $GL(V)$ on $S_\lambda V$ is given by:

$$\chi_{S_\lambda}(g) = \text{trace of the matrix of } g \text{ acting on } S_\lambda.$$

If V has basis $\{v_1, \dots, v_n\}$ then $Sym^2 V$ has basis $v_i \otimes v_j + v_j \otimes v_i$ (i may be equal to j)

And $\begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$ sends $v_i \otimes v_j + v_j \otimes v_i$ to $x_i x_j (v_i \otimes v_j + v_j \otimes v_i)$.

$\therefore$ trace of $g = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$ acting on $S_{(2)} V$ is

$$x_1^2 + x_1 x_2 + x_1 x_3 + \dots + x_{n-1} x_n + x_n^2 = h_2(x_1, \dots, x_n).$$

This is true for a diagonal $g \in GL(V)$ and hence for a diagonalizable $g \in GL(V)$.
<u>Fact:</u> diagonalizable elements are dense inside $GL(V)$.
$\therefore$ this is true for all of $GL(V)$

$\Longrightarrow$

$$\chi_{S_{(2)}}(g) = H_2(x_1, \dots, x_n) \text{ for all } g \in GL(V),$$

where the eigenvalues of $g \in GL(V)$ are $x_1 \ldots x_n$.

Similarly, $\chi_{S_{(d)}}(g) = h_d(x_1, \ldots, x_n)$ for all $g \in GL(V)$.
$= S_d(x_1, \ldots, x_n)$
correspond to semistd fillings of $\square\square\square\cdots\square$

Ex: If $\lambda = (1, 1, \ldots, 1)$ then $S_\lambda V = \sum_{w \in S_d} \operatorname{sgn}(w)\, w\, V^{\otimes d}$

Element of $S_{(1,1)}(V)$ have the form

$$(e_{id} - e_{(12)})(v_i \otimes v_j) = v_i \otimes v_j - v_j \otimes v_i$$

Note: if $i = j$ this element is 0.

So if $\{v_1, \ldots, v_n\}$ is basis for V then

$S_{(1,1)}(V)$ has basis of element $v_i \otimes v_j - v_j \otimes v_i$
where $i \neq j$.

$$g = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$$ sends $v_i \otimes v_j - v_j \otimes v_i$ to
$x_i x_j (v_i \otimes v_j - v_j \otimes v_i)$

So trace of g acting on $S_{(1,1)}(V)$ is
$$x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n = e_2(x_1, \ldots, x_n)$$
$$= S_{(1,1)}(x_1, \ldots, x_n)$$

More generally,

$$\chi_{S_{(1,1,\dots,1)}}(g) = S_{(1,1,\dots,1)}(x_1,\dots,x_n)$$

where the eigenvalues of $g \in GL(V)$ are $x_1, x_2, \dots, x_n$.

Theorem: (1) Let $k = \dim V$. Then $S_\lambda V$ is zero if $\lambda_{k+1} \neq 0$. If $\lambda = (\lambda_1 \geq \dots \geq \lambda_k \geq 0)$ then

$$\dim S_\lambda V = S_\lambda(1,\dots,1) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j-i}$$

(2) $V^{\otimes d} \cong \bigoplus_\lambda (S_\lambda V)^{\otimes f^\lambda}$ where f^λ = dim of the irrep V_λ of S_d

(3) For $g \in GL(V)$, the trace of g on $S_\lambda V$ = value of the Schur poly on the eigenvalues $x_1, \dots, x_k$ of g on V:

$$\chi_{S_\lambda V}(g) = S_\lambda(x_1,\dots,x_k).$$

(4) Each $S_\lambda V$ is an irrep of $GL(V)$.

Before proving this, note that the construction V_λ of irreps of S_d using tabloids can be re-stated in a way analogous to this construction of reps of $GL(V)$.

Recall: V_λ has basis given by $v_T = b_T\{T\}$

$V_\lambda \subset M_\lambda$ (tabloid module)

Claim: Tabloid module $M_\lambda \cong$ the image of a_λ (by right mult) on $\mathbb{C}[S_d]$,

and $V_\lambda \cong$ the image of c_λ (by right multiplication) on $\mathbb{C}[S_d]$

Here $c_\lambda = a_\lambda b_\lambda$ where $a_\lambda = \sum_{w \in Row(\lambda)} w$, $b_\lambda = \sum_{w \in Col(\lambda)} sgn(w)\, w$

Note that for any $u \in S_d$, $u \cdot a_\lambda$ can be identified w/ a tabloid:

Say $\lambda = (3,2)$ $\quad \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array}$

$Row(\lambda) = S_{\{1,2,3\}} \times S_{\{4,5\}}$

Say $u = (4,2,1,5,3)$. ← list notation

$\begin{array}{|c|c|c|} \hline 4 & 2 & 1 \\ \hline 5 & 3 \\ \cline{1-2} \end{array}$ ↙ Depict permutations by placing their values in the boxes of the tableau.

The effect of multiplying u by any element in $Row(\lambda)$ is to permute elemts w/in each row.

$$\text{So } u \cdot a_\lambda = \underbrace{\begin{array}{|c|c|c|} \hline 4 & 2 & 1 \\ \hline 5 & 3 \\ \cline{1-2} \end{array} + \begin{array}{|c|c|c|} \hline 2 & 4 & 1 \\ \hline 5 & 3 \\ \cline{1-2} \end{array} + \cdots + \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}}_{12 \text{ terms}}$$

which we can identify w/ the tabloid $\begin{array}{c} \hline 1\ 2\ 4 \\ \hline 3\ 5 \\ \hline \end{array}$