155 Lecture 16

An overview of what we have done so far

Studied representations of S_n.
Irreducibles $\Leftrightarrow$ conjugacy classes $\Leftrightarrow$ partitions of n.

For $\lambda \vdash n$, defined M^λ, a module whose basis is tabloids of shape λ.
In fact $M^\lambda \simeq \mathrm{Ind}_{S_\lambda}^{S_n} 1$, where

$S_\lambda \cong S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_k}$, the row group of a tableau of shape λ.

M^λ not irreducible.
However, it has a submodule V^λ which is irreducible.
Basis elements for V^λ are of the form $V_T := b_T \cdot \{T\}$ where T is a standard tableau of shape λ.

$\therefore \dim V^\lambda = \#$ standard tableau of shape λ.
This number denoted f^λ & is given by the hook length formula.

For any finite group G, $|G| = \sum (\dim V_i)^2$, where sum is over all irred. reps of G.

$$\therefore n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

We proved this combinatorially, using RSK: Bijection between permutations & pairs of std tableaux of same shape.

How could one have "discovered" tabloid approach to rep theory of S_n?
One way to get new representations from old is to induce representations of subgps of S_n up to S_n. Every group has the trivial representation.

We know we are looking for $p(n)$ irred irreps of S_n, so let's find $p(n)$ different subgps of S_n & induce their trivial reps.
To each partition $\lambda \vdash n$ we have a natural subgp of S_n,
$S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_k}$

permutes 1 to λ_1, permutes λ_1+1 to λ_2, etc.

So consider $\mathrm{Ind}_{S_\lambda}^{S_n} 1 \cong M^\lambda$.

These are not irred so need to find irreps w/in them.
(This requires some cleverness...)

We also gave another approach to rep theory of S_n, different than tabloid approach - Vershik - Okounkov approach.
Idea is to analyze what happens when one restricts an irrep of S_n to S_{n-1} — it breaks into irreps for S_{n-1}.
Repeating procedure, down to S_1, we get a "canonical" decomposition of an irrep of S_n into 1-dim vector spaces V_T.

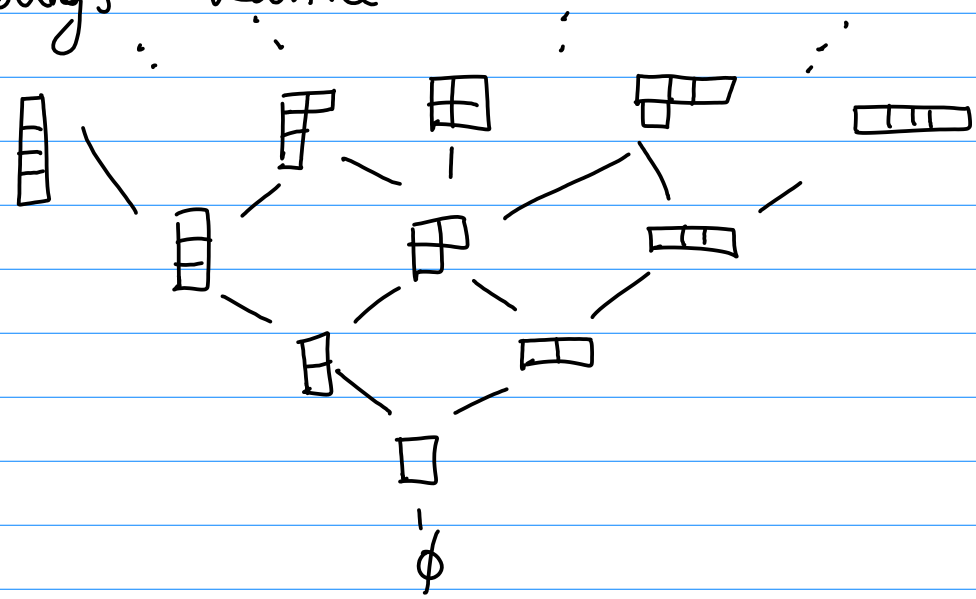
We studied a max'l comm. subalgebra of $\mathbb{C}[S_n]$, generated by the J M elements X_i, $1 \le i \le n$. In the basis given by the V_T, the X_i's are all diagonal.

Analysis of how S_n differs from S_{n-1} & of the eigenvalues of the X_i on the V_T led naturally to the idea of standard tableaux:

If we fix a V_T, the vector of eigenvalues of $(X_1, \dots, X_n)$ on V_T is exactly the content vector of a standard tableaux.

This approach also led naturally to the branching rule for irreps of S_n:
If $\lambda \vdash n$,
$$V^\lambda \Big|_{S_{n-1}} = \bigoplus_\mu V^\mu$$
where μ varies over all partitions obtained from λ by deleting an outer corner.

Young's Lattice:



Note that saturated chains in lattice $\leftrightarrow$ standard tableaux.

There is a combinatorial story that is parallel to the rep theory of S_n: the theory of symmetric functions.

Λ = ring of symmetric functions.

$\Lambda = \bigoplus_{n \geq 0} \Lambda^n$ where Λ^n = homog. symm. functions of degree n.

$\dim \Lambda^n = p(n)$:
Easy to see once we have defined the monomial symmetric functions m_λ.

We studied 5 bases of Λ:
$\{m_\lambda\}$, $\{e_\lambda\}$, $\{h_\lambda\}$, $\{p_\lambda\}$, $\{s_\lambda\}$.

Schur function s_λ the most important.
Can be defined in 3 ways:
combinatorial (semistd tableaux)
classical (ratio of determinants)
determinant in the h's

We typically proved that a set $\{u_\lambda\}_{\lambda \vdash n}$ was a basis of Λ^n by considering the transition matrix between $\{u_\lambda\}$ & another set known to be a basis & showing it was invertible.

Some useful structure on Λ.

- Involution ω, defined by $\omega(e_n) = h_n$.
  Allowed us to prove some things for h's that we already knew about e's.
  As you'll see on next HW, $\omega(s_\lambda) = s_{\lambda'}$, where λ' is the conjugate of λ.

- Inner product $\langle , \rangle$ on Λ.
  This was defined by $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$
  We proved it was symmetric.
  We also proved that $\{u_\lambda\}$, and $\{v_\lambda\}$ are dual bases of Λ iff

$$\sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \quad (*)$$

We then used this to prove that $\{s_\lambda\}$ is an <u>orthonormal</u> basis (dual to itself).
The proof of ⊛ for $\{s_\lambda\}$ again used RSK, this time the version for semistandard tableaux.

We then started to see connections between rep theory of S_n & symmetric functions.

Let ϕ^μ be the char. of M^μ, the (non-irred) module on tabloids of shape λ.
We proved that $p_\lambda = \sum \phi^\mu(\lambda)\, m_\mu$.

This was not so hard: since S_n acts on M^μ by <u>permuting</u> the tabloids, we were able to evaluate the character by counting fixed points.
However, what we <u>really</u> wanted was a connection between the <u>irreps</u> of S_n & symmetric functions.

To do this, we needed to understand how M^μ decomposes into irreps.
This was provided by Young's Rule!

$$M^\mu \cong \bigoplus_\lambda K_{\lambda\mu} V^\lambda, \text{ where}$$

$K_{\lambda\mu}$ = # semistd tableaux of shape λ & content μ.

This implies that $\langle \phi^\mu, \chi^\lambda \rangle = K_{\lambda\mu}$.

The main step to prove Young's Rule is Pieri's Rule!

Theorem: The representation $V \otimes 1$ appears in $\operatorname{Res}^{S_n}_{S_{n-k}\times S_k} V^\nu$ once if ν is a k-expansion of λ, and zero otherwise. Equivalently,
$$\operatorname{Ind}^{S_n}_{S_{n-k}\times S_k}(V_\lambda \otimes 1) = \bigoplus_\nu V_\nu$$
where sum is over all k-expansions ν of λ.

We proved this by using GT basis & analyzing the restriction of V^ν.

Once we knew Young's Rule $\left(M^\mu \cong \bigoplus_\lambda K_{\lambda\mu} V^\lambda\right)$ we were able to use

$p_\lambda = \sum \phi^\mu(\lambda)\, m_\mu$ to prove that

$s_\lambda = \frac{1}{n!} \sum_{\pi \in S_n} \chi^\lambda(\pi) p_\pi$. We also rewrote this as

$$s_\lambda = \sum_\mu z_\mu^{-1} \chi^\lambda(\mu) p_\mu . \quad (*)$$

(What is z_μ? Write $\mu = (1^{m_1} 2^{m_2} \ldots n^{m_n})$. Then $z_\mu = 1^{m_1} m_1! \cdot 2^{m_2} m_2! \ldots n^{m_n} m_n!$.)

Let R^n = ring of class functions on S_n.
$\dim R^n = p(n)$.

Inspired by ⊛, we made the following definition:
Def: The characteristic map is $ch^n : R^n \to \Lambda^n$ defined by
$$ch^n(\chi) = \sum_{\mu \vdash n} z_\mu^{-1} \chi(\mu)\, p_\mu$$

Then $ch^n(\chi^\lambda) = s_\lambda$ — we are mapping irreps of S_n to the Schur functions.

Recall: We have inner prod $\langle\,,\rangle$ on class functions too, s.t. irred. char's form orthonormal basis.

Prop: The linear transformation ch^n is a bijection s.t. $\langle \phi, \chi \rangle = \langle ch^n(\phi), ch^n(\chi) \rangle$ for any $\phi, \chi \in R^n$.

So far just have a map $R^n \to \Lambda^n$.
Would like a map $? \to \Lambda = \bigoplus_n \Lambda^n$

? should be $\bigoplus R^n$.
Note: there is a natural product on Λ — can multiply symm functions, & this is "compatible w/ the grading":

i.e. if $f \in \Lambda^n$ and $g \in \Lambda^m$ then $fg \in \Lambda^{n+m}$.

What is the product on $R = \bigoplus_n R^n$?

If eg $\chi \in R^n$ a char. of S_n and $\psi \in R^m$ a character of S_m, need to define a "product" of χ and ψ which is in R^{m+n} — hopefully a character of S_{m+n}.

Most natural way to do this is:
$$\chi \cdot \psi = \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (\chi \otimes \psi)$$

Then if we define $ch = \bigoplus_n ch^n$, get:

<u>Theorem</u>: The map $ch : R \to \Lambda$ is a bijective ring homomorphism:

it satisfies $ch(f \cdot g) = (ch f)(ch g)$

We now have a kind of "dictionary" between rep theory of S_n & symm. functions, via the map ch.

$$\xrightarrow{ch}$$

$R = \bigoplus_n R^n$	$\Lambda = \bigoplus_n \Lambda^n$
χ^λ is ortho. basis of R^n w/ respect to $\langle , \rangle$	s_λ is ortho. basis of Λ^n w/ respect to $\langle , \rangle$
$\chi^{(n)}$ is trivial rep	$s_{(n)} = h_n$
$\chi^{(1^n)}$ is alternating rep	$s_{(1^n)} = e_n$
Product of χ_μ & χ_ν is $\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} \chi_\mu \otimes \chi_\nu$	Product of symm. functions f & g is multiplication fg.
Pieri Rule: $\mathrm{Ind}_{S_{n-k} \times S_k}^{S_n} (V_\lambda \otimes 1) = \bigoplus_\nu V_\nu$ where sum over all k-expansions ν of λ	Pieri Rule: If $\lambda \vdash n-k$, $s_\lambda h_k = \sum_\nu s_\nu$ where sum over all k-expansions ν of λ.
Young's Rule: $M^\mu \cong \bigoplus_\lambda K_{\lambda\mu} V^\lambda$, $M^\mu = \mathrm{Ind}_{S_\lambda}^{S_n} 1$	$h_\lambda = \sum K_{\lambda\mu} s_\lambda$

Multiplying χ^λ by sign rep.

Question: How does $\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} \chi_\mu \otimes \chi_\nu$ decompose into irreps for S_{m+n}? What are #'s $c_{\mu\nu}^\lambda$ s.t.

$$\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} = \bigoplus c_{\mu\nu}^\lambda V_\lambda \ ?$$

Involution ω.

What are #'s $c_{\mu\nu}^\lambda$ s.t.

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda \ ?$$

With this connection between symm. functions + tableaux combinatorics to rep theory of S_n, we can use the combinatorics to answer rep theory questions. The application we saw last time was the Murnaghan-Nakayama Rule:

<u>Theorem:</u> $$\chi^\lambda(\alpha) = \sum_T (-1)^{ht(T)}$$

Summed over all border strip tableaux of shape λ/μ + type α.

We proved this using the fact that $s_\lambda = \sum_\nu z_\nu^{-1} \chi^\lambda(\nu) p_\mu$, by proving that

$s_\lambda = \sum_\nu z_\nu^{-1} \psi^\lambda(\nu) p_\mu$, where $\psi^\lambda(\nu) =$ # border strip tableaux of shape λ & type ν.

Main step was showing: for $\mu \in \mathrm{Par}$ and $r \in \mathbb{N}$,

$$s_\mu p_r = \sum_\lambda (-1)^{ht(\lambda/\mu)} s_\lambda,$$

summed over all partitions $\lambda \supseteq \mu$ for which λ/μ is a border strip of size r.

As a tangent, we also used RSK to prove more facts about permutations.

<u>Theorem:</u> (Schensted) Consider $\pi \in S_n$. The length of the longest increasing subsequence of π = length of first row of $P(\pi)$! Length of longest decreasing " " = length of first column of $P(\pi)$.

Noting that this only takes into account first rows + columns of $P(\pi)$, we went on to prove a generalization of this:

Theorem (Green): Given $\pi \in S_n$, let $\text{sh } P(\pi) = (\lambda_1, \dots, \lambda_\ell)$ w/ conjugate $(\lambda_1', \dots, \lambda_m')$. Then for any k,

$$i_k(\pi) = \lambda_1 + \dots + \lambda_k,$$
$$d_k(\pi) = \lambda_1' + \dots + \lambda_k'.$$

Def: Let π be a sequence. A subsequence σ of π is <u>k-increasing</u> if, as a set, it can be written as the disjoint union

$\sigma = \sigma_1 \cup^* \sigma_2 \cup^* \dots \cup^* \sigma_k$ where the σ_i

are increasing subsequences of π. If the σ_i are all decreasing, call σ <u>k-decreasing</u>.

Let $i_k(\pi)$ = length of π's longest k-increasing subsequences.

$d_k(\pi)$ = " " " k-decreasing subsequences.