155 Lecture 14

Last time: observed that the S_n-module M^λ (basis given by tabloids) is $\cong_{S_n}$

$$\operatorname{ind}_{R(T)}^{S_n} 1 = \operatorname{ind}_{S_{\lambda_1}\times S_{\lambda_2}\times \dots S_{\lambda_k}}^{S_n} 1 .$$

Write S_λ for $S_{\lambda_1}\times S_{\lambda_2}\times \dots \times S_{\lambda_k}$.

<u>Def:</u> Call ν a <u>k-expansion</u> of λ if Young diagram of ν is obtained from that of λ by adding k boxes in different columns.

We proved:

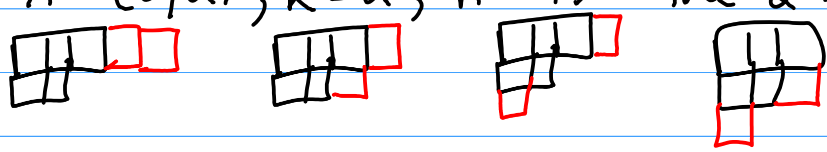
(Pieri's Rule) Let $\lambda \vdash n-k$

<u>Theorem:</u> The representation $V^\lambda \otimes 1$ appears in $\operatorname{Res}^{S_n}_{S_{n-k}\times S_k} V^\nu$ once if ν is a k-expansion of λ, and zero otherwise. Equivalently,

$$\operatorname{Ind}^{S_n}_{S_{n-k}\times S_k}(V_\lambda \otimes 1) = \bigoplus_\nu V_\nu$$

where sum is over all k-expansions ν of λ.

<u>Example:</u> Say $\lambda=(3,2)$, $k=2$, $n=7$. The 2-expansions of λ are:



So

$$\operatorname{ind}^{S_7}_{S_5\times S_2}(V_\lambda\otimes 1) = V_{(5,2)} \oplus V_{(4,3)} \oplus V_{(4,2,1)} \oplus V_{(3,3,1)}$$

Cor: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$.
Then there is one copy of V_μ in $\mathrm{Ind}_{S_\lambda}^{S_n} \mathbb{1}$
for each chain
$\emptyset = \nu_0 < \nu_1 < \ldots < \nu_r = \mu$ of partitions s.t. for each
$1 \le i \le r$, ν_i is a (λ_i)-expansion of ν_{i-1}.

<u>Proof</u>: Induction. When $r=1$, only possibility for μ is $\mu = (n)$. (row).
So $\nu_r = \nu_1 = (n)$ and $\lambda = (n)$.
Need that there is one copy of $V_{(n)}$ (= trivial)
in $\mathrm{Ind}_{S_\lambda}^{S_n} \mathbb{1} = \mathrm{Ind}_{S_n}^{S_n} \mathbb{1}$. ✓

Pieri's Rule says
$$\mathrm{Ind}_{S_a \times S_b}^{S_{a+b}} (V_\lambda \otimes 1) = \bigoplus_\nu V_\nu$$ where sum is over
all b-expansions ν of λ. $(\lambda \vdash a)$

Now note that to compute $\mathrm{Ind}_{S_a \times S_b \times S_c}^{S_{a+b+c}} \mathbb{1} = \mathrm{Ind}_{S_a \times S_b \times S_c}^{S_{a+b+c}} 1 \otimes 1 \otimes 1$,
it's equivalent to compute
$$\mathrm{Ind}_{(S_{a+b}) \times S_c}^{S_{a+b+c}} \left[\left(\mathrm{Ind}_{S_a \times S_b}^{S_{a+b}} 1 \otimes 1 \right) \otimes 1 \right] = \mathrm{Ind}_{(S_{a+b}) \times S_c}^{S_{a+b+c}} \left[\left(\bigoplus_\nu V_\nu \right) \otimes 1 \right]$$
acts trivially

ν: b-expansion of the partition (a)

By Pieri $=$ $$\bigoplus_\mu V_\mu .$$
μ a c-expansion of ν where ν a b-expansion of (a)

Continuing in this fashion, corollary follows.

But these chains $\phi = \nu_0 < \nu_1 < \ldots < \nu_r = \mu \longleftrightarrow$ semistd tableaux of shape μ.

<u>Cor:</u> There is one copy of V_μ in $\mathrm{Ind}_{S_\lambda}^{S_n} 1$ for each semistd Young tableau of shape μ containing λ_i squares labeled i for each i.

Since $M^\lambda \cong \mathrm{ind}_{S_\lambda}^{S_n} 1$, this shows:

<u>Theorem: (Young's Rule)</u>

$$M^\lambda \cong \bigoplus_\mu K_{\mu\lambda} V^\mu$$

Recall power sym. functions:

$p_n = \sum_i x_i^n$ and $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$.

If π has cycle type λ, let p_π denote p_λ (abuse of notation).

If ϕ^μ denotes the character of M^μ, then we showed last time that

$\star$ $p_\pi = \sum_\mu \phi^\mu(\pi) m_\mu$.

Now we want to relate χ^λ, the char of V^λ, to symm. functions.

Use $\star$ and Young's Rule.

To bring in χ^λ and inner product, multiply $\star$ by $\frac{\chi^\lambda(\pi)}{n!}$ and sum, to get:

$$\frac{1}{n!}\sum_{\pi\in S_n} p_\pi \chi^\lambda(\pi) = \frac{1}{n!}\sum_{\pi\in S_n}\left(\sum_\mu \phi^\mu(\pi) m_\mu\right)\chi^\lambda(\pi)$$

$$= \sum_\mu m_\mu \cdot \frac{1}{n!}\sum_\pi \phi^\mu(\pi)\chi^\lambda(\pi)$$

$$= \sum_\mu m_\mu \langle \phi^\mu, \chi^\lambda\rangle$$ ASK! (Young's Rule)

$$= \sum_\mu K_{\lambda\mu} m_\mu$$ ← ASK!

$$= s_\lambda.$$

<u>Cor</u>: If $\lambda \vdash n$, $s_\lambda = \frac{1}{n!}\sum_{\pi\in S_n}\chi^\lambda(\pi)p_\pi$.

s_λ is a sort of "generating function" for the values of the character χ^λ.

Other way to write the character...

Since χ^λ a class function, can collect terms:

$$s_\lambda = \frac{1}{n!} \sum_\mu k_\mu \chi^\lambda_\mu p_\mu \quad \text{where } k_\mu = |K_\mu| \text{ conj. class}$$

and χ^λ_μ = value of χ^λ on K_μ.

Since $\frac{n!}{k_\mu}$ = size of centralizer of μ, we can rewrite again.

Write $\mu = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$.

Define $z_\mu := 1^{m_1} m_1!\, 2^{m_2} m_2! \ldots n^{m_n} m_n!$

z_μ = size of centralizer of elemt. in class K_μ.

So $s_\lambda = \sum_\mu \frac{1}{z_\mu} \chi^\lambda_\mu p_\mu$. ⊛

Let $R^n = R(S_n)$ = space of class functions on S_n. Then $\dim R^n = p(n)$.

<u>Recall:</u> we have an inner prod on R^n for which the irred. chars of S_n form orthonormal basis.

Recall def. of inner prod $\langle \phi, \chi \rangle = \frac{1}{n!} \sum_{\pi \in S_n} \phi(\pi)\chi(\pi)$.

Also, $\dim \Lambda^n = p(n)$, and we have an inner prod on Λ^n for which the Schur functions form orthonormal basis.

Def: The characteristic map is $ch^n : R^n \to \Lambda^n$ defined by

$$ch^n(\chi) = \sum_{\mu \vdash n} z_\mu^{-1} \chi(\mu)\, p_\mu$$

where $\chi(\mu)$ is the value of χ on the class μ.

Equivalently, defining the class function ψ by $\psi(w) = p_{\rho(w)}$ ← cycle type of w, we have $ch^n(\chi) = \langle \chi, \psi \rangle$. $\oplus$

(this is obviously equiv. to the corollary we had)

By $\circledast$, $ch^n(\chi^\lambda) = s_\lambda$.

This gives the explicit map between class functions on S_n & symm. functions.

Prop: The linear transformation ch^n is an isometry, ie. $\langle \phi, \chi \rangle = \langle ch^n(\phi), ch^n(\chi) \rangle$ for any $\phi, \chi \in R^n$.

Proof: ch^n takes one orthonormal basis to another. ∎

Let $R = \bigoplus_n R^n$, which is isomorphic to $\Lambda = \bigoplus_n \Lambda^n$ as vector spaces via the map

$$ch = \bigoplus_n ch^n$$

We want a product on class functions that corresponds via ch^n to the usual product on symmetric functions.

If χ and ψ are char's of S_n & S_m, respectively, we want to produce a char. of S_{n+m}.
But tensor prod gives us a char of $S_n \times S_m$, not S_{n+m}. So we need to induce.

Define prod on R by bilinearly extending

$$\chi \cdot \psi = \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (\chi \otimes \psi)$$

Theorem: The map $ch : R \to \Lambda$ is a bijective ring homomorphism:

It satisfies $ch(f \cdot g) = (ch f)(ch g)$

(S_m permutes $1 \dots m$, S_n permutes $m+1 \dots m+n$)

Proof:

$$\mathrm{ch}(f\cdot g) = \mathrm{ch}\left(\mathrm{Ind}_{S_m\times S_n}^{S_{m+n}} (f\otimes g)\right) \qquad \text{By } \oplus$$

$$= \left\langle \mathrm{Ind}_{S_m\times S_n}^{S_{m+n}} f\otimes g,\ \psi \right\rangle \qquad \text{Frobenius Reciprocity}$$

$$= \left\langle f\otimes g,\ \mathrm{Res}_{S_m\times S_n}^{S_{m+n}} \psi \right\rangle_{S_m\times S_n}$$

$$= \frac{1}{m!\,n!} \sum_{u\in S_m} \sum_{v\in S_n} f(u)g(v)\,\psi(uv)$$

$$= \frac{1}{m!\,n!} \sum_{u\in S_m} \sum_{v\in S_n} f(u)g(v)\,\psi(u)\psi(v)$$

$$= \langle f,\psi\rangle_{S_m} \langle g,\psi\rangle_{S_n}$$

$$= (\mathrm{ch}\, f)(\mathrm{ch}\, g). \qquad \blacksquare$$

Obs: Since $\mathrm{ch}(\chi^{\lambda}) = s_{\lambda}$, we have $\mathrm{ch}\left(\chi^{\underbrace{\square\square\square\square\square}_{n}}\right) = \mathrm{ch}(\text{trivial}) = s_{\square\square\square\square\square}$

$$= h_{(n)}.$$

And $\mathrm{ch}(\chi^{(1^n)}) = \mathrm{ch}(\text{sign rep}) = s_{(1^n)} = e_{(n)}$.

Suppose we try to compute the expansion

$$s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^{\lambda} s_\lambda \quad (c_{\mu\nu}^{\lambda} \text{ coefficients}).$$

What rep theory problem does this correspond to?

Say $\mu \vdash m$, $\nu \vdash n$,

Then $s_\mu s_\nu = \mathrm{ch}(\chi_\mu \chi_\nu) = \mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} \chi_\mu \otimes \chi_\nu$

and $s_\nu = \mathrm{ch}(\chi_\nu)$.

So we are asking how the representation $\mathrm{Ind}_{S_m \times S_n}^{S_{m+n}} V_\mu \otimes V_\nu$ decomposes into irreps.

The numbers $c_{\mu\nu}^{\lambda}$ are called the Littlewood Richardson numbers.

<u>Def</u>: A <u>ballot sequence</u> or <u>lattice permutation</u> is a sequence of positive integers $\pi = i_1 i_2 \ldots i_n$ s.t., for any prefix $\pi_k = i_1 i_2 \ldots i_k$

and any positive integer l, the # of l's in π_k is at least as large as the number of $(l+1)$'s in that prefix.

A reverse ballot sequence or lattice perm. is a sequence π s.t. π^r is a ballot sequence.

<u>Theorem</u> (Littlewood Richardson Rule) The value of the coefficient $c^{\lambda}_{\mu\nu}$ is equal to the number of semistandard tableau T s.t.

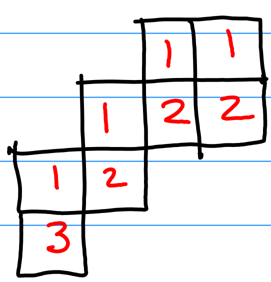
1. T has shape λ/μ & content ν
2. The row word of T, π_T, is a reverse lattice perm.

Recall row word is $R_l R_{l-1} \dots R_1$.
So reverse of row word is $R_1^r R_2^r \dots R_l^r$

<u>Example</u>: $\lambda = (4,4,2,1)$, $\mu = (2,1)$, $\nu = (4,3,1)$.
What is $c^{\lambda}_{\mu\nu}$?

1's 2's 3's

(2)



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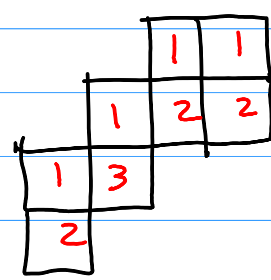

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