155 Lecture 13 (connection to rep theory of S_n)

Recall: have scalar product on Λ, defined by requiring that
$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu} \quad \text{for all } \lambda, \mu \in \mathrm{Par}.$$

Lemma: Let $\{u_\lambda\}$ and $\{v_\lambda\}$ be bases of Λ s.t. for all $\lambda \vdash n$, $u_\lambda, v_\lambda \in \Lambda^n$. Then $\{u_\lambda\}$, and $\{v_\lambda\}$ are dual bases iff
$$\sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$$

Proof: Write $m_\lambda = \sum_\rho \zeta_{\lambda\rho} u_\rho$ and $h_\mu = \sum_\nu \eta_{\mu\nu} v_\nu$.

So $\delta_{\lambda\mu} = \langle m_\lambda, h_\mu \rangle = \sum_{\rho, \nu} \zeta_{\lambda\rho} \eta_{\mu\nu} \langle u_\rho, v_\nu \rangle$. $\quad (\star)$

For fixed $n \geq 0$, regard ζ and η as matrices indexed by $\mathrm{Par}(n)$ & let A be the matrix defined by $A_{\rho\nu} = \langle u_\rho, v_\nu \rangle$.

So $(\star)$ is equivalent to $\quad \xleftarrow{\text{transpose}}$ (always true)
$$I = \zeta A \eta^t.$$

So $\{u_\lambda\}$ and $\{v_\mu\}$ are dual bases $\iff$
$$A = I \iff$$
$$I = \zeta \eta^t \iff$$
$$I = \eta \zeta^t \iff$$
$$\delta_{\rho\nu} = \sum_\lambda \zeta_{\lambda\rho} \eta_{\lambda\nu}$$

But we saw earlier that

$$\prod_{i,j} (1-x_i y_j)^{-1} = \sum_\lambda m_\lambda(x) h_\lambda(y)$$

$$= \sum_\lambda \Big(\sum_\rho \varsigma_{\lambda\rho} u_\rho(x) \Big) \Big(\sum_\nu \eta_{\lambda\nu} v_\nu(y) \Big)$$

$$= \sum_{\rho,\nu} \underbrace{\Big(\sum_\lambda \varsigma_{\lambda\rho} \eta_{\lambda\nu} \Big)} u_\rho(x) v_\nu(y).$$

If this $= \delta_{\rho\nu}$ then get

$$= \sum_\rho u_\rho(x) v_\rho(y)$$

So dual bases $\Rightarrow$ identity holds

And if identity $\prod (1-x_i y_j)^{-1} = \sum_\rho u_\rho(x) v_\rho(y)$, then since the power series $u_\rho(x) v_\nu(y)$ are lin. indep., get

$$\sum_\lambda \varsigma_{\lambda\rho} \eta_{\lambda\nu} = \delta_{\rho\nu} \Rightarrow \text{dual bases.}$$

Recall that we had:

<u>Corollary</u>: Let $X = \{x_1, x_2, \ldots\}$, $Y = \{y_1, y_2, \ldots\}$

Then
$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_{\lambda \in Par} s_\lambda(x) s_\lambda(y)$$

$\therefore$ by Lemma, $\{s_\lambda\}$ is a dual basis to itself.

M^λ not necess. irred. Contains the irred S^λ.

Cor: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.

Now: Connection to rep theory of S_n.

Recall that if $\lambda \vdash n$, M^λ is the v.s. whose basis is set of all tabloids of shape λ. S_n acts on M^λ by permuting the tabloids. Recall that in any perm. rep, $\chi(g) = \#$ of elemts of set fixed by g

Ex: $n = 3$. $\lambda = (3), (2,1),$ or $(1,1,1)$.
Denote character of M^λ by ϕ^λ and conj. class of S_3 corresp. to μ by K_μ.
Conj. class rep's: id, (12), (123)

$\lambda = (3)$: one tabloid $\boxed{1\ 2\ 3}$. trivial rep.

$\lambda = (2,1)$: tabloids are $\begin{array}{l}\hline 1\ 2 \\ \hline 3 \\ \hline\end{array}$ $\begin{array}{l}\hline 1\ 3 \\ \hline 2 \\ \hline\end{array}$ $\begin{array}{l}\hline 2\ 3 \\ \hline 1 \\ \hline\end{array}$

e fixes all 3, (12) fixes 1, (123) fixes none

$\lambda = (1,1,1)$: tabloids are $\begin{array}{c}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$ $\begin{array}{c}\hline 1 \\ \hline 3 \\ \hline 2 \\ \hline\end{array}$ $\dots$ $\begin{array}{c}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}$ (six)

e fixes all 6, (12) and (123) fix none

	$K_{(1^3)}$	$K_{(2,1)}$	$K_{(3)}$
$\phi^{(3)}$	1	1	1
$\phi^{(2,1)}$	3	1	0
$\phi^{(1^3)}$	6	0	0

Last basis of symm. functions: power-sum

Def: The n^{th} power sum symm. function is
$p_n = m_{(n)} = \sum_{i \geq 1} x_i^n$.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, then
$p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$.

Example: For $\lambda \vdash 3$, expand p_λ into m_μ's...

$p_{(3)} = x_1^3 + x_2^3 + \ldots \quad = \quad m_{(3)}$

$p_{(2,1)} = (x_1^2 + x_2^2 + \ldots)(x_1 + x_2 + \ldots) \quad =$ ASK!

What monomials can appear here?

$x_1^3 + x_2^3 + \ldots \quad (= m_{(3)})$

$x_1^2 x_2 + x_1^2 x_3 + \ldots + x_2^2 x_1 + x_2^2 x_3 + \ldots \quad (= m_{(2,1)})$

So $p_{(2,1)} = m_{(3)} + m_{(2,1)}$

$p_{(1,1,1)} = (x_1 + x_2 + x_3 + \ldots)^3 \quad = \quad ?$

$x_1^3 + x_2^3 + \ldots \quad (= m_{(3)})$

$3(x_1^2 x_2 + x_1^2 x_3 + \ldots + x_2^2 x_1 + x_2^2 x_3 + \ldots) \quad (= 3m_{(2,1)})$

$6(x_1 x_2 x_3 + x_1 x_2 x_4 + \ldots) \quad (= 6m_{(1,1,1)})$

So $p_{(1,1,1)} = m_{(3)} + 3m_{(2,1)} + 6m_{(1,1,1)}$

Compare w/ the table computed earlier!

Theorem: Let ϕ_λ^μ be the character of M^μ evaluated on class corresp. to λ. Then $p_\lambda = \sum_{\mu \trianglerighteq \lambda} \phi_\lambda^\mu m_\mu$.

Write $p_\lambda = \prod_i (x_1^{\lambda_i} + x_2^{\lambda_i} + \dots) = \sum_\mu c_{\lambda\mu} m_\mu$.

Note: $c_{\lambda\mu} = 0$ unless $\mu \trianglerighteq \lambda$ (dominance order).
Because if $x_1^{\mu_1} x_2^{\mu_2} \dots x_m^{\mu_m}$ appears in $p_\lambda = (x_1^{\lambda_1} + x_2^{\lambda_1} + \dots)(x_1^{\lambda_2} + x_2^{\lambda_2} + \dots)$ then each μ_i must be a sum of λ_j's.
Since adding together parts of a partition makes it larger in dominance order, $\mu \trianglerighteq \lambda$.

Proof: Consider coeff of x^μ on both sides.
On LHS, it is # of ways to distribute the parts of λ into subpartitions $\lambda^1, \dots, \lambda^m$ s.t. $\bigcup_i^* \lambda^i = \lambda$ and $\lambda^i \vdash \mu_i \ \forall i$

(*) where equal parts of λ are distinguished in order to be considered different in disjoint union.

<u>Ex:</u> $\mu = (4, 2)$, $\lambda = (\overset{\#1}{2}, \overset{\#2}{2}, 1, 1)$

$\lambda^1 = (2,2)$ and $\lambda^2 = (1,1)$. $(\underline{x_1^2} + x_2^2 + \dots)(\underline{x_1^2} + x_2^2 + \dots)(x_1^1 + \underline{x_2^1} + \dots)(x_1^1 + \underline{x_2^1} + \dots)$

OR

$\lambda^1 = (\overset{\#1}{2}, 1, 1)$ and $\lambda^2 = (\overset{\#2}{2})$ $(\underline{x_1^2} + x_2^2 + \dots)(x_1^2 + \underline{x_2^2} + \dots)(\underline{x_1^1} + x_2 + \dots)(\underline{x_1^1} + x_2^1 + \dots)$

OR

$\lambda^1 = (\underset{\#2}{2}, 1, 1)$ and $\lambda^2 = (\underset{\#1}{2})$ $(x_1^2 + \underline{x_2^2} + \dots)(\underline{x_1^2} + x_2^2 + \dots)(\underline{x_1^1} + x_2 + \dots)(\underline{x_1^1} + x_2^1 + \dots)$

Now consider $\phi^\mu_\lambda = \phi^\mu(\pi)$ where π has cycle type λ.
By def., this character value $=$ # of fixed points of the action of π on all std tabloids t of shape μ.

But t is fixed iff each cycle of π lies in a single row of t. So need to distribute the cycles of length λ_i among the rows of μ_j exactly according to $(*)$. $\therefore$ $c_{\lambda\mu} = \phi^{\mu}_{\lambda}$.

Recall: the irrep V^λ is a subrep of M^λ.

Want to relate symm. functions to V^λ.

Need to know how M^λ decomposes into irreps.
Recall $K_{\lambda\mu}$ = # semistd tableaux of shape λ and content μ.

<u>Theorem: (Young's Rule)</u>

$$M^\mu \cong \bigoplus_\lambda K_{\lambda\mu} V^\lambda$$

In particular, if $\chi^\lambda = \text{char } V^\lambda$,
$$\langle M^\mu, \chi^\lambda \rangle = K_{\lambda\mu}.$$

Once we know this, we can try to relate symm. functions to character χ^λ of V^λ by using inner product.

(skip the def. ?!)

maybe just talk about $\mathrm{ind}_H^G 1$ + Frobenius Reciprocity

Recall the idea of inducing representations:
Let $H \subset G$ be a subgp and let W be a representation of H.
We will construct a rep $V := \mathrm{ind}_H^G W$ of G:

- Choose a representative $g_\sigma \in G$ for each coset $\sigma \in G/H$, w/ e representing trivial coset H.
- Take a copy W^σ of W for each left coset $\sigma \in G/H$.
- For $w \in W$, let $g_\sigma w$ denote the element of W^σ corresp. to $w \in W$.
- Let $V = \bigoplus_{\sigma \in G/H} W^\sigma$, so each element of V has a unique expression $v = \sum_\sigma g_\sigma w_\sigma$ for elements $w_\sigma \in W$.

- Given $g \in G$, define
$$g \cdot (g_\sigma w_\sigma) = g_\tau (h w_\sigma) \text{ if } g \cdot g_\sigma = g_\tau \cdot h.$$
for some $\tau \in G/H$ and $h \in H$

Easy example: Consider $V = \mathrm{ind}_H^G 1$. trivial
W = one-dim trivial rep.
V has dimension $|G/H|$.
Denote a basis by $\{e_\sigma : \sigma \in G/H\}$.
$g.(e_\sigma\} = e_\tau$ where
$g.\sigma = \tau . h$ for $h \in H \Rightarrow$
τ is just the coset of $g\sigma$.
So $g.e_\sigma = e_{g\sigma}$

So this is just the perm. rep. on cosets of G/H.

<u>Observation:</u> Let $\lambda \vdash n$ and let T be the standard tableau filled as

$$\begin{array}{|l|}\hline 1\ 2\ \dots\ \qquad\qquad\qquad \lambda_1 \\ \hline \lambda_1+1 \dots \qquad \lambda_1+\lambda_2 \\ \hline \qquad \text{etc.} \\ \hline \dots \quad \Sigma\lambda_i \\ \hline\end{array}$$

Let $R(T)$ be the row group of S_n.
$(R(T) \cong S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_k})$

Then $$M^\lambda \cong \operatorname{ind}_{R(T)}^{S_n} 1 \quad \left(\cong \operatorname{ind}_{S_{\lambda_1} \times \dots \times S_{\lambda_k}}^{S_n} 1 \right)$$

Each tabloid represents a coset in $S_n/R(T)$, and M^λ is just the permutation representation on the tabloids.

<u>Frobenius Reciprocity:</u> Let $H \leq G$ and suppose ψ and χ are char's of H & G, resp. Then
$$\langle \operatorname{ind}_H^G \psi, \chi \rangle = \langle \psi, \operatorname{Res}_H^G \chi \rangle.$$

<u>Def:</u> Call ν a <u>k-expansion</u> of λ if Young diagram of ν is obtained from that of λ by adding k boxes in different columns.

(main tool to prove Young's Rule)

(Pieri's Rule) Let $\lambda \vdash n-k$

Theorem: The representation $V^\lambda \otimes 1$ appears in $\mathrm{Res}^{S_n}_{S_{n-k}\times S_k} V^\nu$ once if ν is a k-expansion of λ, and zero otherwise. Equivalently,

$$\mathrm{Ind}^{S_n}_{S_{n-k}\times S_k}(V_\lambda \otimes 1) = \bigoplus_\nu V_\nu$$ where sum is over all k-expansions ν of λ.

We will use GT basis here.

Recall the GT basis is defined up to choice of scalar.

There is in fact a nice choice of scalar s.t. the following holds:

Prop: Let T be a std tableau. If i and $i+1$ are horiz. adj. in T, then $s_i . v_T = v_T$. If i and $i+1$ are vertically adj. in T, $s_i . v_T = -v_T$. If T and T' are related by admissible transposition s_i (i and $i+1$ in diff. rows & diff. columns), the subspace of vectors in $\mathbb{C}v_T \oplus \mathbb{C}v_{T'}$ fixed by s_i is spanned by $v_T + v_{T'}$.

(We proved the part about $s_i v_T = \pm v_T$, & the second part follows easily from analysis of $H(2)$)

Now let's prove Theorem.

Proof: Need to find # of copies of $V^\lambda \otimes 1$ in $\mathrm{Res}^{S_n}_{S_{n-k}\times S_k} V^\nu$. Here $S_{n-k} = S_{\{1,\dots,n-k\}}$ and $S_k = S_{\{n-k+1,\dots,n\}}$

Let W be subspace of V^ν obtained by restricting to S_{n-k} + taking components isomorphic to V^λ.
For T a standard tableau of shape ν,
let $t(T)$ = union of boxes labeled by $1 \dots n-k$.
(smaller tableau)

Branching Rule $\Rightarrow$ W = span of the GT basis of shape ν vectors v_T where T ranges over all tableaux w/ $t(T)$ of shape λ.
So $\dim W = \dim V^\lambda$ times # of ways to fill in squares of λ/ν w/ $n-k+1, \dots, n$ s.t. rows + columns are increasing.

$W \subset V^\nu$ and as rep of S_{n-k}, $W = \bigoplus V^\lambda$.
Since $S_{n-k} \times S_k$ acts on V^ν, so does $\{1\} \times S_k$.
Need to find # of times the trivial rep of $\{1\} \times S_k$ occurs in W, ie. the dim. of subspace of W fixed by all of S_k.

Use the GT basis v_T. Fix std Young tableau U of shape λ and set

$$W_U = \bigoplus_{t(T)=U} V_T.$$

Then W_U is a $\{1\} \times S_k$-submodule since action of $\{1\} \times S_k$ preserves the entries $1, \dots, n-k$ in each tableau! S_k only permutes entries $n-k+1 \dots n$.

By Prop, a vector $v = \sum_{t(T)=u} c_T v_T$ in W_u is fixed by $\{1\} \times S_k$ iff:

(1.) For each T in which i and $i+1$ are <u>vertically</u> adj. for $i \geq n-k+1$, $c_T = 0$.

(2.) For each T and T' w/ $t(T) = t(T') = u$ related by an admissible transposition $(i \;\; i+1)$ for $i \geq n-k+1$, we have $c_T = c_{T'}$.

In fact if $t(T) = t(T')$, we can connect T and T' by a sequence of admissible transpositions in $\{1\} \times S_k$. So (2.) $\Rightarrow$ <u>all</u> coeffs c_T are equal.

So the subspace of W_u fixed by all of $\{1\} \times S_k$ is contained in span of $v_0 = \sum_{t(T)=u} v_T$

(i.e. 0 or 1-dimensional).

Condition (1) $\Rightarrow$ v_0 is fixed by all of $\{1\} \times S_k$ iff there is no tableau T w/ $t(T) = u$ w/ i and $i+1$ vertically adjacent for some $i \geq n-k+1$.

In other words, the k boxes added to Young diagram of λ to form ν are in diff columns, i.e. ν a k-expansion of λ. $\blacksquare$