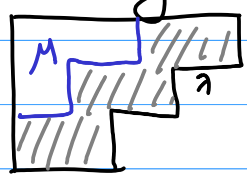
155 Lecture 12

Def: If λ is a partition and $\mu \subset \lambda$ (containment of Young diagrams), the <u>skew shape</u> λ/μ is:

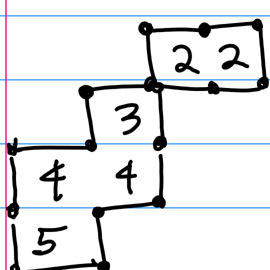


← the shaded region.

Def: A semistd tableau of skew shape λ/μ is a filling of the boxes of λ/μ w/ numbers $\{T_{ij}\}$ s.t. rows weakly increase & columns strictly increase.

Given a filling, define $\mu_k = \#\{T_{ij} = k\}$. Then $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ is called the <u>type</u> or <u>content</u> of T.

Ex: $\lambda = (4,2,2,1)$, $\mu = (2,1)$, filling as follows:



This has shape λ/μ and type $(0,2,1,2,1)$.

If T an SST of type μ we associate the monomial

$$X^T = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}.$$

So here we associate $x_2^2 x_3 x_4^2 x_5$

Another combinatorial object in bijection w/ SST: Gelfand–Tsetlin patterns.

A GT-pattern is a triangular array G of nonnegative integers, say

$$\begin{array}{ccccccccc}
a_{11} & & a_{12} & & a_{13} & \cdots & & & a_{1n} \\
& a_{22} & & a_{23} & & \cdots & & a_{2n} & \\
& & a_{33} & \cdots & & & a_{3n} & & \\
& & & \ddots & & \iddots & & & \\
& & & & a_{nn} & & & &
\end{array}$$

s.t. $a_{ij} \le a_{i+1,j+1} \le a_{i,j+1}$ when all 3 numbers are defined.

So rows weakly increase and $a_{i+1,j+1}$ is between its upper 2 neighbors.

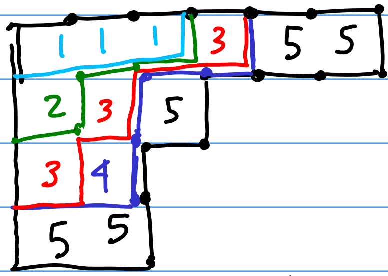
eg

$$\begin{array}{ccccccccc}
0 & & 2 & & 2 & & 3 & & 6 \\
& 0 & & 2 & & 2 & & 4 & \\
& & 1 & & 2 & & 4 & & \\
& & & 1 & & 3 & & & \\
& & & & 3 & & & &
\end{array}$$

These are in bijection w/ SST!

Let λ^i be i^{th} row of G in reverse order. Define tableau $T = T(G)$ by inserting $n-i+1$ into squares of the skew shape $\lambda^i / \lambda^{i+1}$.

Here, the sequence of skew shapes is:



w/ filling

A connection w/ representation theory:

In fact the Okounkov–Vershik approach to rep theory of S_n is analogous to a classical approach to rep theory of GL_n.

Irred. polynomial reps V^λ of GL_n $\Longleftrightarrow$ partitions λ w/ at most n rows

Basis of V^λ indexed by semistandard tableaux of shape λ filled with numbers $\leq n$.

These irreps V^λ have simple branching when restricted to GL_{n-1} ... so can get canonical basis ... & when we write down the eigenvalues of certain generators of max'l comm. subgp of GL_n on basis elements, we get these GT patterns.

The character of V^λ is the Schur polynomial $S_\lambda(x_1, \ldots, x_n)$!

Back to Schur functions ... recall definition.

Let $T(n, \lambda/\mu)$ = set of SST of shape λ/μ filled w/ entries $\leq n$. $T(\lambda/\mu)$ = set of all SST of shape λ.

<u>Def</u>: Let λ/μ be a skew shape.
Then $$s_{\lambda/\mu}(x_1, x_2, \ldots) = \sum_{T \in T(\lambda/\mu)} x^T$$ ← Schur function.

Note $$s_{\lambda/\mu}(x_1, x_2, \ldots, x_n) = \sum_{T \in T(n, \lambda/\mu)} x^T$$ ← Schur polynomial

When $\mu = \phi$, we get a standard shape.

Let $\lambda \vdash n$ and α a weak composition of n.

<u>Def</u>: The Kostka number $K_{\lambda\alpha}$ is: the # of SST of shape λ and content α

So $s_\lambda(x) = \sum K_{\lambda\alpha} x^\alpha$.

Prop: $s_{\lambda/\mu}(x)$ is symmetric.

Proof: Enough to show $(i\ i{+}1)\, s_{\lambda/\mu}(x) = s_{\lambda/\mu}(x)$ for each adjacent transposition.

Plan: define an involution on SST of shape λ/μ $\quad T \to T'$

s.t. the numbers of i's and $(i+1)$'s are exchanged when go from T to T'.

Given T, each column contains one of following:

- an $i, i+1$ pair
- exactly one of i and $i+1$
- neither

Call the pairs <u>fixed</u> and all other occurrences of i and $i+1$ <u>free</u>.

In each row switch the number of free i's and $(i+1)$'s: i.e. if row has k free i's followed by l free $(i+1)$'s then replace by l free i's followed by k free $(i+1)$'s:

If $i=2$ and

pair free

$$T = \begin{array}{llllllllll} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 3 \\ 2 & 2 & 3 & 3 & 3 & 3 \\ 3 \end{array}$$

pair free pair

type $(4,7,6)$

then

$$T' = \begin{array}{llllllllll} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\ 2 & 2 & 2 & 3 & 3 & 3 \\ 3 \end{array}$$

type $(4,6,7)$

New tableau T' still semistd.
Since fixed i's and $(i+1)$'s come in pairs, map has desired exchange property.
Also is clearly an involution. ▨

Cor: $s_\lambda = \sum_\mu K_{\lambda\mu} m_\mu$.

As you'll see in the homework, there are other ways to define s_λ.

Th: Let $\lambda = (\lambda_1, .., \lambda_n) \in \mathrm{Par}$ where we allow possibility of 0's at end i.e. $\lambda_{i+1} = \lambda_{i+2} = \cdots = \lambda_n = 0$.

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det\left(x_j^{n-i+\lambda_i}\right)_{i,j=1}^{n}}{\prod_{1 \le i < j \le n} (x_i - x_j)}$$

Theorem: Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition w/ possible 0's at the end. Then

$$s_\lambda(x_1, \dots, x_n) = \det\left(h_{\lambda_i - i + j}\right)_{i,j=1}^{n}$$

where we set $h_k = 0$ for $k < 0$.

Ex: $s_{(4,1)}(x_1, x_2, x_3) =$

$\lambda = (4,1,0)$

$$\det\begin{pmatrix} h_{4-1+1} & h_{4-1+2} & h_{4-1+3} \\ h_{1-2+1} & h_{1-2+2} & h_{1-2+3} \\ \underbrace{h_{0-3+1}}_{=0} & \underbrace{h_{0-3+2}}_{=0} & h_{0-3+3} \end{pmatrix}$$

Theorem: The set $\{s_\lambda : \lambda \in Par(n)\}$ is a basis for $\Lambda^n(X)$, $X = \{x_1, \dots, x_n\}$.

This follows from

Prop: Suppose μ & λ are partitions w/ $|\mu| = |\lambda|$ and $K_{\lambda\mu} \neq 0$. Then $\mu \leq \lambda$ in dominance order. Also, $K_{\lambda\lambda} = 1$.

Proof: Recall dominance order, i.e. $\mu \leq \lambda$ iff $\forall i$, $\mu_1 + \dots + \mu_i \leq \lambda_1 + \dots + \lambda_i$.

Suppose $K_{\lambda\mu} \neq 0$. By def, $\exists$ a SST T of shape λ and type μ.

Suppose a part $T_{ij} = k$ appears below the

k^{th} row (ie. $i > k$). Then
$$1 \leq T_{1k} < T_{2k} < \dots < T_{ik} = k \text{ for } i > k, \Rightarrow\Leftarrow.$$

So the parts $1, 2, \dots, k$ all appear in the first k rows, so $\mu_1 + \mu_2 + \dots + \mu_k \leq \lambda_1 + \lambda_2 + \dots + \lambda_k$.
$\therefore \quad \mu \leq \lambda$.

And if $\mu = \lambda$, we must have $T_{ij} = i$
$\forall \ (i,j)$,
$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \cline{1-4} 3 & 3 & 3 \\ \cline{1-3}\end{array} \qquad \text{so } K_{\lambda\lambda} = 1. \qquad \blacksquare$$

Now using dominance order, we have
$$\begin{pmatrix} s_\lambda \end{pmatrix} = \begin{pmatrix} & K_{\lambda\mu} & \end{pmatrix} \begin{pmatrix} m_\mu \end{pmatrix} \quad \text{where}$$

upper Δ w/ 1's on diag $\Rightarrow$ invertible

$$\Rightarrow \quad \begin{pmatrix} m_\mu \end{pmatrix} = \begin{pmatrix} K_{\lambda\mu} \end{pmatrix}^{-1} \begin{pmatrix} s_\lambda \end{pmatrix}$$

$\Rightarrow \quad \{ s_\lambda \mid \lambda \vdash n \}$ spans $\Lambda^n(X)$.
Since $\dim \Lambda^n(X) = \# \mathrm{Par}(n)$,

$\{ s_\lambda \mid \lambda \vdash n \}$ is a basis for $\Lambda^n(X)$.

Applications of RSK to Symmetric functions:

Recall that more general version of RSK gives bijection from matrices $A = (a_{ij})_{i,j \geq 1}$ over $\mathbb{N}$ w/ finitely many nonzero elements to pairs (P,Q) of SST of same shape w/ $n = \sum_{i,j} a_{ij}$ boxes. Moreover, type$(P) = $ row(A), type$(Q) = $ col(A).

Corollary: Let $X = \{x_1, x_2, \dots\}$, $Y = \{y_1, y_2, \dots\}$. Then

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_{\lambda \in Par} s_\lambda(x) s_\lambda(y)$$

Proof: Write

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \prod_{i,j} (1 + x_i y_j + x_i^2 y_j^2 + \dots) = \prod_{i,j} \left[\sum_{a_{ij} \geq 0} (x_i y_j)^{a_{ij}} \right] \quad (*)$$

What is the meaning of the RHS? A monomial $x^\alpha y^\beta = (x_1^{\alpha_1} x_2^{\alpha_2} \dots)(y_1^{\beta_1} y_2^{\beta_2} \dots)$ the RHS corresponds to a choice of a natural # a_{ij} for each pair i,j, ie. a matrix A whose ij entry is a_{ij}.

$$\text{row}(A) = \left(\sum_{j \geq 1} a_{1j}, \sum a_{2j}, \dots\right)$$

$$= (\alpha_1, \alpha_2, \dots)$$

$$\text{col}(A) = (\beta_1, \beta_2, \dots)$$

So coeff of $x^\alpha y^\beta$ in $\circledast$ = # matrices A w/ entries in $\mathbb{N}$ w/ $\mathrm{row}(A) = \alpha$, $\mathrm{col}(A) = \beta$.

But by combinatorial interpretation of Schur functions, the coeff of $x^\alpha y^\beta$ in $\sum_\lambda s_\lambda(x) s_\lambda(y)$ = # ordered pairs (P, Q) of SST's of same shape with $\mathrm{type}(P) = \alpha$, $\mathrm{type}(Q) = \beta$.
Result follows by RSK correspondence. ∎

<u>Now</u>: define a scalar product on Λ, i.e. a bilinear form $\Lambda \times \Lambda \longrightarrow \mathbb{Q}$, denoted $\langle \, , \rangle$.

If $\{u_i\}$ and $\{v_j\}$ are bases of a v.s. V, a scalar product is uniquely determined by specifying values of $\langle u_i, v_j \rangle$ $\forall i, j$.

Say $\{u_i\}$ and $\{v_j\}$ are <u>dual bases</u> if $\langle u_i, v_j \rangle = \delta_{ij}$.

<u>Def</u>: Define $\langle \, , \rangle$ on Λ by requiring that $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$ for all $\lambda, \mu \in \mathrm{Par}$.

<u>Note</u>: If f and g are homogeneous, $\langle f, g \rangle = 0$ unless $\deg f = \deg g$.

<u>Prop</u>: $\langle \, , \rangle$ is symm., i.e. $\langle f, g \rangle = \langle g, f \rangle$ $\forall f, g \in \Lambda$.

Proof: Enough to show this is true for the elements of a basis. Use $\{h_\lambda\}$.

$\langle h_\lambda, h_\mu \rangle = \langle \sum_\nu N_{\lambda\nu} m_\nu, h_\mu \rangle = N_{\lambda\mu}$.

Since $N_{\lambda\mu} = N_{\mu\lambda}$, $\langle h_\lambda, h_\mu \rangle = \langle h_\mu, h_\lambda \rangle$. ✓

Now: a tool to verify orthogonality of certain classes of symm. functions.

Lemma: Let $\{u_\lambda\}$ and $\{v_\lambda\}$ be bases of Λ s.t. for all $\lambda \vdash n$, $u_\lambda, v_\lambda \in \Lambda^n$. Then $\{u_\lambda\}$, and $\{v_\lambda\}$ are dual bases iff

$$\sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$$

Recall that we had:

Corollary: Let $X = \{x_1, x_2, \ldots\}$, $Y = \{y_1, y_2, \ldots\}$.

Then $\displaystyle\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda \in Par} s_\lambda(x) s_\lambda(y)$

$\therefore$ by Lemma, $\{s_\lambda\}$ is a dual basis to itself.

Cor: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.