# 155 Lecture 11

<u>Def:</u> Let $X = \{x_1, x_2, x_3, \ldots\}$ be finite or countable. $f(x_1, x_2, \ldots) = \sum_\alpha c_\alpha x^\alpha$ is a symmetric function over $X$ if:

ⓐ $f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots)$ over all permutations of $\mathbb{P}$.

ⓑ The monomials appearing in $f$ have bounded degree.

<u>Note:</u> we really can say all permutations here.

ⓐ just means that if the coeff of $x_1^{\alpha_1} x_2^{\alpha_2} \ldots$ in $f(x_1, \ldots)$ is $c_\alpha$, then for any perm $\sigma$ of $\mathbb{P}$, the coeff of $x_{\sigma(1)}^{\alpha_1} x_{\sigma(2)}^{\alpha_2} \ldots$ is also $c_\alpha$.

Doesn't matter if 2 perms $\alpha$ and $\tilde{\alpha}$ send the monomial $x_1^{\alpha_1} x_2^{\alpha_2}$ to the same monomial.

$\Lambda(X)$ — v.s. of all symm functions

$\Lambda^m(X)$ — v.s. of homog. deg $m$ symm functions

<u>monomial symmetric function</u>

$m_\lambda(x) = \sum_\alpha x^\alpha$ where $\alpha$ runs through all <u>distinct</u> permutations of $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_r, 0, 0, \ldots)$

(Here we do need to be careful)

Clearly $\{m_\lambda : \lambda \vdash m\}$ a basis for $\Lambda^m(X)$

<u>element symm. functions</u>: $e_0(x) = 1$,

$e_k(x) = \sum_{i_1 < \ldots < i_k} x_{i_1} \ldots x_{i_k}$, $e_\lambda(x) = \prod e_{\lambda_i}(x)$

Recall: for matrix $A = (a_{ij})_{i,j \geq 1}$ w/ finitely many non-zero elemts,
$\text{row}(A) := (r_1, r_2, \ldots)$ the <u>row sum vector</u>
where $r_i = \sum_j a_{ij}$.
Similarly define $\text{col}(A)$.

<u>Prop</u>: Let $\lambda = (\lambda_1, \lambda_2, \ldots) \in \text{Par}(m)$. Then

$$e_\lambda = \sum_{\mu \in \text{Par}(m)} M_{\lambda\mu} m_\mu$$

where $M_{\lambda\mu}$ = # of 0-1 matrices $A = (a_{ij})_{i,j \geq 1}$ w/ $\text{row}(A) = \lambda$, $\text{col}(A) = \mu$.

<u>Proof</u>: We have

$$e_\lambda = e_{\lambda_1} \cdot e_{\lambda_2} \cdot \ldots = \sum_{i_1 < i_2 < \ldots < i_{\lambda_1}} x_{i_1} x_{i_2} \ldots x_{i_{\lambda_1}} \cdot \sum_{j_1 < j_2 < \ldots < j_{\lambda_2}} x_{j_1} x_{j_2} \ldots x_{j_{\lambda_2}} \cdot \ldots$$

We can represent any product $x_{i_1} \ldots x_{i_{\lambda_1}} \cdot x_{j_1} \ldots x_{j_{\lambda_2}}$ that appears in the sum on the RHS as a 0-1 matrix by writing 1's in columns $i_1, \ldots, i_{\lambda_1}$ of row 1, in columns $j_1, \ldots, j_{\lambda_2}$ of row 2, etc. (filling up A's w/ 0's).
So $\text{row}(A) = \lambda$.
This product is a monomial $x_1^{\mu_1} x_2^{\mu_2} \cdots$ iff $x_i$ appears $\mu_i$ times, i.e. $\text{col}(A) = \mu$. $\blacksquare$

<u>Ex</u>: $\lambda = (3,1,1)$. The matrices $A$ w/ $\text{row}(A) = \lambda$, $\text{col}(A) = \mu$ are:

$\mu = (3,1,1)$

$$\begin{array}{c|c} 3 & \\ 1 & \\ 1 & \\ & \end{array}\begin{array}{c} \begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 1 & 1 \end{matrix} \end{array}$$

$\mu = 221$

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 2 & 1 \end{matrix}$$

Save this picture for later or point out pushing 1's to left

$\mu = (2,1,1,1)$

$$\begin{array}{c|cccc} 3 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ \hline & 2 & 1 & 1 & 1 \end{array}$$

can permute the 2nd, 3rd, 4th column in 3 ways. For each of these 3, can also exchange 2nd + 3rd rows

$2 \cdot 3 = \underline{6}$ diff matrices here

$\boxed{\text{also}}$

$$\begin{array}{c|cccc} 3 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline & 2 & 1 & 1 & 1 \end{array} \quad +1 \qquad 6+1=7$$

$\mu = (1,1,1,1,1)$

$$\begin{array}{c|ccccc} 3 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ \hline & 1 & 1 & 1 & 1 & 1 \end{array}$$

Can get 20 matrices from these: permute columns in $\binom{5}{3}$ ways, then permute the two rows containing only one 1 in 2 ways. $\binom{5}{3} \cdot 2 = 20$.

$$\therefore \; e_{311} = m_{311} + 2m_{221} + 7m_{2111} + 20m_{11111}.$$

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**Remark:** Since

$M_{\lambda\mu}$ = # of $0-1$ matrices $A = (a_{ij})_{i,j \geq 1}$ w/ $\text{row}(A) = \lambda$, $\text{col}(A) = \mu$,

by considering the transpose of $A$, it's clear that $M_{\lambda\mu} = M_{\mu\lambda}$ for any $\lambda, \mu \in \text{Par}(n)$.

Fund. Theorem of Symm. Functions:
The set $\{e_\lambda(x) : \lambda \in Par(m)\}$ is a basis for $\Lambda^m(X)$.

Pf: We know from the fact that monomial symm. functions $\{m_\lambda : \lambda \in Par(m)\}$ form basis for $\Lambda^m(x)$ that $\dim \Lambda^m(x) = p(m)$.
So to show that $\{e_\lambda(x) : \lambda \in Par(m)\}$ is a basis, enough to show they span $\Lambda^m(x)$.
That is, enough to show that any $m_\mu$ $(\mu \in Par(m))$ is in the span of $\{e_\lambda(x) : \lambda \in Par(m)\}$.

Idea: We know how to express e's in terms of m's, i.e.
$$e_\lambda = \sum_{\mu \in Par(m)} M_{\lambda\mu} m_\mu$$

This gives transition matrix between e's and m's, i.e.
$$\begin{pmatrix} e_\lambda \end{pmatrix} = \begin{pmatrix} M_{\lambda\mu} \end{pmatrix} \begin{pmatrix} m_\mu \end{pmatrix}$$
where rows + columns are indexed by partitions of m (in some order). If we can show that after ordering partitions of m in some way, $(M_{\lambda\mu})$ is upper triangular with 1's on diagonal, then $(M_{\lambda\mu})$ is invertible $\Rightarrow$
$(m_\mu) = (M_{\lambda\mu})^{-1} (e_\lambda)$, i.e. we can express $m_\mu$'s in terms of $e_\lambda$'s.

First: define total ordering on set $\mathrm{Par}(m)$.
Set $\lambda < \mu \iff \lambda_i < \mu_i$ where $i$ is the first index w/ $\lambda_i \neq \mu_i$.
This is the lexicographic order on $\mathrm{Par}(m)$.
Eg. For $m=5$, $11111 < 2111 < 221 < 311 < 32 < 41 < 5$

Claim: (a) If $M_{\lambda\mu} > 0$, then $\mu \leq \lambda^*$ where $\lambda^*$ is conjugate to $\lambda$.
(b) $M_{\lambda\lambda^*} = 1$.
This will show that we can write $(M_{\lambda\mu})$ as an upper $\Delta$ matrix w/ 1's on diagonal →

$$
\begin{array}{cc|ccccc}
 & & 5 & 41 & 32 & 311 & \cdots \\
\hline
 & 11111 & 1 & * & * & * & \\
 & 2111 & 0 & 1 & * & * & \cdots \\
\lambda & 221 & 0 & 0 & 1 & * & * \\
 & 311 & 0 & 0 & 0 & 1 & \\
 & 32 & 0 & 0 & 0 & 0 & 1 \\
 & 41 & 0 & 0 & 0 & 0 & 0 \\
 & 5 & & & & & \\
\end{array}
$$

To prove claim:

If $M_{\lambda\mu} > 0$, then $\exists$ 0 1 matrix $A$ w/ $\mathrm{row}(A) = \lambda$, $\mathrm{col}(A) = \mu$.
By def. of lex order, we obtain the maximal possible $\mu$ w/ $M_{\lambda\mu} > 0$ by pushing all 1's in each row to the left!
(See old example:) !
But this gives exactly the Young diagram for the partition $\lambda$ (w/ a 1 in each box), with $\lambda_i$ 1's in each row.
Reading instead the columns as the partition $\mu$, this is $\lambda^*$.
$\therefore$ $\mu = \lambda^*$ is max possible $\mu$, and in this case we have $M_{\lambda\lambda^*} = 1$. $\blacksquare$

## Complete symm. Functions

We get natural analogue to element. symm. functions $e_k(x)$ by allowing repetitions of the indices.

Define complete symm. functions $h_k$ by

$$h_k(x_1, x_2, \ldots) = \sum_{i_1 \le i_2 \le \ldots \le i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \qquad (k \ge 1)$$

$$h_0(x_1, x_2, \ldots) = 1.$$

Note: $h_k(x) = \sum_{\mu \in Par(k)} m_\mu(x)$ and $h_k(x) \in \Lambda^k(x)$

As for e's, define

$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_r} \in \Lambda^m(X)$.

Obs: $\sum_{k \ge 0} h_k(x_1, x_2, \ldots) z^k = \prod_i \frac{1}{1 - x_i z}$ (#1)

Clear by expanding RHS

Compare to $\sum_{k \ge 0} e_k(x_1, x_2, \ldots) z^k = \prod_i (1 + x_i z)$ (#2)

Analog to previous prop:

Prop: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in Par(m)$. Then

$$h_\lambda = \sum_{\lambda \in Par(m)} N_{\lambda\mu} m_\mu \qquad \text{where}$$

$N_{\lambda\mu} = \#$ matrices $A = (a_{ij})_{i,j \ge 1}$ w/ entries which are natural #'s (includes 0)

with $\text{row}(A) = \lambda$, $\text{col}(A) = \mu$.
Further, $N_{\lambda\mu} = N_{\mu\lambda}$.

<u>Proof:</u> Exercise. This is easy – take transpose of $A$.

<u>Theorem:</u> The set $\{h_\lambda(x) : \lambda \in \text{Par}(m)\}$ is a basis for $\Lambda^m(X)$.

<u>Proof:</u> Since $\{e_\lambda : \lambda \in \text{Par}(m)\}$ is a basis, suffices to show any $e_\lambda$ in span of $\{h_\mu : \mu \in \text{Par}(m)\}$.
By #1 and #2, $\left(\sum_{k\geq 1} e_k z^k\right)\left(\sum_{k\geq 1} h_k (-z)^k\right) = 1$

Looking at coeff of $z^n$ above,

#3 $\displaystyle\sum_{k=0}^{n} e_k \cdot (-1)^{n-k} \cdot h_{n-k} = 0$ for $n \geq 1$.

Now $e_0 = h_0 = 1$, $e_1 = h_1$, so assume $n \geq 2$.
By #3 and induction on $n$

$$e_n = -\sum_{k=0}^{n-1} e_k (-1)^{n-k} h_{n-k} \in \langle h_{j_1} \cdots h_{j_t} : \textstyle\sum j_i = n \rangle$$

$\therefore$ $e_n$ is in span of the $h_\lambda$'s $(\lambda \vdash n)$
$\Rightarrow$ for $\lambda = (\lambda_1, \ldots, \lambda_r) \in \text{Par}(m)$,

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_r} \in \langle h_{j_1} \cdots h_{j_\ell} : \textstyle\sum j_i = m \rangle = \langle h_\mu : \mu \in \text{Par}(m) \rangle$$

$\boxed{\text{PM}}$

Ex: Consider $\lambda = 311$. By (A3),

$e_1 = h_1$, $\quad e_2 = e_1 h_1 - e_0 h_2 = h_1^2 - h_2$,

$$e_3 = e_2 h_1 - e_1 h_2 + e_0 h_3 = (h_1^2 - h_2) h_1 - h_1 h_2 + h_3$$
$$= h_1^3 - 2h_1 h_2 + h_3.$$

Thus $e_{311} = e_3 e_1 e_1 = (h_1^3 - 2h_1 h_2 + h_3) h_1^2$
$$= h_1^5 - 2h_1^3 h_2 + h_1^2 h_3.$$

So $e_{311} = h_{11111} - 2h_{2111} + h_{311}$.

Prop: $$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda,\mu} N_{\lambda\mu} m_\lambda(x) m_\mu(y)$$
$$= \sum_\lambda m_\lambda(x) h_\lambda(y)$$
where $\lambda$ and $\mu$ range over Par.

Proof: Use previous result that
$$h_\lambda = \sum_{\mu \vdash n} N_{\lambda\mu} m_\mu.$$

Interpreting LHS, we get

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \prod_{i,j} (1 + x_i y_j + x_i^2 y_j^2 + \dots) = \prod_{i,j} \Big[ \sum_{a_{ij} \geq 0} (x_i y_j)^{a_{ij}} \Big] \quad (*)$$

What is the meaning of $x^\alpha y^\beta = (x_1^{\alpha_1} x_2^{\alpha_2} \dots)(y_1^{\beta_1} y_2^{\beta_2} \dots)$ here → A monomial corresponds to a choice of a natural # $a_{ij}$ for each pair $i,j$, ie. a matrix $A$ whose $ij$ entry is $a_{ij}$. We have:

$$\mathrm{row}(A) = \left(\sum_{j\geq 1} a_{1j}, \sum a_{2j}, \ldots\right)$$
$$= (\lambda_1, \lambda_2, \ldots)$$
$$\mathrm{col}(A) = (\mu_1, \mu_2, \ldots)$$

So coeff of $x^\lambda y^\mu$ on LHS $= N_{\lambda\mu}$, the # of $\mathbb{N}$-matrices $A = (a_{ij})$ where $\mathrm{row}(A) = \lambda$, $\mathrm{col}(A) = \mu$.

This corresponds to the RHS.

Close relation of $e_n$ & $h_n$:

Define $\omega : \Lambda \to \Lambda$ by $\omega(e_n) = h_n$ $(n \geq 0)$ & extend to algebra homomorphism, i.e. define $\omega(e_\lambda) = h_\lambda$ for any $\lambda$.

By (3), $0 = \sum_{k=0}^{n} h_k (-1)^{n-k} \omega(h_{n-k})$

$$= \sum_{l=0}^{n} \omega(h_l)(-1)^{n-l} h_{n-l} \qquad \text{defining } l := n-k$$

But then since by (3), $\sum_{l=0}^{n} e_l (-1)^{n-l} h_{n-l}$,

it follows that $\omega(h_l) = e_l$ $\forall\, l$.

$\therefore$ $\omega$ is an involution, i.e. $\omega^2 = 1$.

<u>Note</u>: When $X = \{x_1, \ldots, x_n\}$ is finite, we can regard this as special case of symm function in $\infty$ variables where we set $x_i = 0$ for $i \geq n+1$.

# Now: Schur Functions

**Def:** A semi standard tableau $T$ (SST) of shape $\lambda$ is a filling of the boxes of the Young diagram w/ pos. integers

$$\begin{matrix} T_{11} & T_{12} & \cdots & T_{1\lambda_1} \\ T_{21} & T_{22} & \cdots & T_{2\lambda} \\ & \vdots & & \\ T_{r1} & T_{r2} & \cdots & T_{r\lambda} \end{matrix}$$

such that rows weakly increase & columns strictly increase,

i.e. $T_{i_1} \leq T_{i_2} \leq \dots \leq T_{i\lambda_i}$ for rows, and $T_{1j} < T_{2j} < \dots$ for columns.

If $\mu_k = \#\{T_{ij} = k\}$ then $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ is called the type or content of $T$.

Example:

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 3 | 3 | 4 |
| 2 | 3 | 4 | | |
| 3 | 5 | 5 | | |
| 4 | 6 | | | |
| 5 | | | | |
| 6 | | | | |

is a SST of shape $(5,3,3,2,1,1)$ and type $(2,1,4,3,3,2)$.

To an SST of type $\mu$ we associate the monomial

$$X^T = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}.$$

Let $T(n,\lambda)$ = set of SST of shape $\lambda$ filled w/ entries $\leq n$. $T(\lambda)$ = set of all SST of shape $\lambda$.

**Def:** Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition, $r \leq n$.
Then $$s_\lambda(x_1, \dots, x_n) = \sum_{T \in T(n,\lambda)} x^T$$

$$s_\lambda(x_1, \dots) = \sum_{T \in T(\lambda)} x^T .$$

Ex: $n=3$, $\lambda = 21$. The SST are:

$$\begin{array}{cccccccc} 1\,1 & 1\,1 & 1\,2 & 1\,2 & 1\,3 & 1\,3 & 2\,2 & 2\,3 \\ 2 & 3 & 2 & 3 & 2 & 3 & 3 & 3 \\ x_1^2x_2 & x_1^2x_3 & x_1x_2^2 & x_1x_2x_3 & x_1x_2x_3 & x_1x_3^2 & x_2^2x_3 & x_2x_3^2 \end{array}$$

$$s_\lambda(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + 2x_1x_2x_3$$

**Note:** $s_{(n)}(x) = h_n(x)$

$s_{(1^n)} = e_n(x)$

"coeff of $x_1x_2\dots x_n$"

If $\lambda \vdash n$ arbitrary, $[x_1x_2\dots x_n]\, s_\lambda(x) = f^\lambda$

**Def:** The Kostka number $K_{\lambda\mu}$ is the # of SST of shape $\lambda$ and content $\mu$.

So $$s_\lambda = \sum_\mu K_{\lambda\mu} x^\mu .$$