155 Lecture 10

<u>Recall:</u> • X_i — Jucy Murphy elemts.
• If v is in the GT basis, let $\alpha(v) = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ be the eigenvalues of $X_1, \ldots, X_n$ on v.

• $\mathrm{Spec}(n) = \{\alpha(v) \mid v \text{ is in the GT basis}\}$
• Write v_α for elemt of GT basis s.t. eigenvalues of X_i's on v_α are given by α.

• $\alpha \sim \beta$ if v_α & v_β in same irrep of S_n.

• $\mathrm{Cont}(n) = \left\{ \begin{array}{c} \text{content vectors of std tableaux} \\ \text{with } n \text{ boxes} \end{array} \right\}$

<u>Last time:</u> $\mathrm{Spec}(n) \subset \mathrm{Cont}(n)$

Now define another equiv. relation.

If $\alpha, \beta \in \mathbb{C}^n$, write $\alpha \approx \beta$ if β is obtained from α by applying a bunch of admissible transpositions — ie. can switch a_i and a_{i+1} as long as $a_{i+1} \neq a_i \pm 1$.

Idea: show $\mathrm{Spec}(n) = \mathrm{Cont}(n)$ and $\sim$ same as $\approx$

Will just sketch the rest of the proof of main result:

#1. Recall that we map a tableau $T = \nu_0 \nearrow \dots \nearrow \nu_n \in Tab(n)$ to $T \mapsto (c(\nu_1 - \nu_0), \dots, c(\nu_n - \nu_{n-1}))$.

Here we are in Young's lattice

We have $\alpha \approx \beta$ for $\alpha, \beta \in Cont(n)$ iff the corresponding paths have the same end, i.e. iff the 2 tableaux have same diagram.

#2. Any two std tableaux $T_1, T_2 \in Tab(\nu)$ w/ diagram ν can be obtained from each other "by a sequence of admissible transpositions.

i.e. if $\alpha, \beta \in Cont(n)$ and $\alpha \approx \beta$ then β can be obtained from α by admiss. transpositions.

(In terms of tableau, admissible transpositions are transpositions of numbers from diff rows & columns — that way content differs by more than 1)

If $a_{i+1} = a_i \pm 1$, ie if content of boxes i & $i+1$ differ by 1, boxes are $\begin{array}{|c|}\hline i \\ \hline i+1 \\ \hline\end{array}$ or $\begin{array}{|c|c|}\hline i & i+1 \\ \hline\end{array}$

$\therefore$ $Cont(n)/\approx$ is the set of classes of std tableaux w/ same diagram.

$\therefore$ $\#\{Cont(n)/\approx\} = p(n)$.

#3. #2 above $\Rightarrow$ if $\alpha \in Spec(n)$ and $\alpha \approx \beta$, for $\beta \in Cont(n)$, then $\beta \in Spec(n)$ and $\alpha \sim \beta$.

because acting by admissible transpositions gives more elements of $Spec(n)$

So each equiv. class in $Cont(n)/\approx$ either has no elements of $Spec(n)/\sim$ or it is a subset of a class in $Spec(n)/\sim$.

But $\#\{Spec(n)/\sim\} = \#\{\hat{S_n}\} = p(n)$ since this is # conj. classes of S_n.

Also we know $\circledast$ $Spec(n) \subset Cont(n)$

Both Spec(n) and Cont(n) have same # equiv classes, so each class in Cont(n) is contained in a class of Spec(n), but now

$\circledast \Rightarrow$ Spec(n) = Cont(n) & the $\sim$ and $\approx$ are the same relation.

#2 $\Rightarrow$ two basis element v_T and $v_{T'}$ are in the same irrep of S_n iff T and T' are tableaux of the same shape. Cont(T) determines the action of the X_i on v_T.

Eg if $\lambda = \begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}$ then two std tableaux

$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ and $T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$. Content $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 & 0 \\ \hline\end{array}$

$\alpha(T_1) = (0,1,-1,0)$, $\alpha(T_2) = (0,-1,1,0)$

$X_1 = 0$, $X_2 = (12)$, $X_3 = (13)+(23)$, $X_4 = (14)+(24)+(34)$

So $X_2 V_{T_1} = V_{T_1}$, $X_3 V_{T_1} = -V_{T_1}$, $X_4 V_{T_1} = 0$

$X_2 V_{T_2} = -V_{T_2}$, $X_3 V_{T_2} = V_{T_2}$, $X_4 V_{T_2} = 0$

It is more useful if we know how <u>all</u> of S_n acts on the V_T.

Let $T \in Tab(\lambda)$ and let $T' = s_i T$ where s_i is an admissible transposition (switches numbers in diff rows + columns)

<u>Prop:</u> We can choose V_T (remember, we have a choice of scalar to make) s.t. for any Coxeter generator s_i and T' defined as above,

$$s_i . V_T = V_{T'} + \frac{1}{(a_{i+1} - a_i)} V_T \quad \text{and}$$

$$s_i . V_{T'} = \left(1 - \frac{1}{(a_{i+1} - a_i)^2}\right) V_T - \frac{1}{(a_{i+1} - a_i)} V_{T'}.$$

Since the s_i generate S_n, this allows us to compute how any $\pi \in S_n$ acts.

<u>Pf:</u> read it!

Now: Symmetric functions

This is an algebraic framework for combinatorial problems.

<u>motivation</u>: Rep theory of S_n, GL_n, Cohom of G/P

<u>Def</u>: A polynomial $f(x_1, \ldots, x_n)$ over $\mathbb{C}$ (or any field of char 0) is called <u>symmetric</u> if

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$$

for all permutations $\sigma \in S_n$.

$f(x_1, \ldots, x_n)$ is <u>alternating</u> if $f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = -f(x_1, \ldots, x_n)$ for all <u>transpositions</u> $\sigma \in S_n$.

<u>Ex</u>: $\prod_{1 \le i < j \le n} (x_i - x_j)$ is an alternating function.

f has <u>degree</u> d if d is the highest degree of the monomials $c x_1^{i_1} \cdots x_n^{i_n}$, $d = \sum_{j=1}^{n} i_j$ appearing in f.

f is <u>homogeneous</u> of degree d if all monomials have degree d.

Example: Classical example is the **elementary symmetric** poly's.

Let $f(x) = \sum_{k=0}^{n} a_k x^{n-k}$ be a poly w/ roots

$x_1, \ldots, x_n$ and leading coeff 1.

We have $f(x) = (x-x_1)(x-x_2)\ldots(x-x_n)$ $\therefore$

$$a_1 = -(x_1 + \ldots + x_n)$$

$$a_2 = x_1x_2 + x_1x_3 + \ldots x_1x_n + \ldots \quad x_{n-1}x_n$$

$$\vdots$$

$$a_k = (-1)^k \sum_{i_1 < \ldots < i_k} x_{i_1} \ldots x_{i_k}$$

$$\vdots$$

$$a_n = (-1)^n x_1 x_2 \ldots x_n .$$

Def: Let $X = \{x_1, x_2, \ldots, x_n\}$. The k^{th} elementary symm. polynomial over X is

$$e_k(x_1, x_2, \ldots, x_n) = \sum_{i_1 < \ldots < i_k} x_{i_1} x_{i_2} \ldots x_{i_k} \quad (k \geq 1)$$

$$e_0(x_1, \ldots, x_n) = 1$$

Clearly e_k is a homogeneous symm. poly of degree k

Def: A <u>monomial</u> over X is any expression $c_\alpha x^\alpha = c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots$ where c_α a constant, $\alpha_i \geq i$ $\forall i$, & all but finitely many α_i are 0. <u>Degree</u> of $c_\alpha x^\alpha$ is $\sum_{i \geq 1} \alpha_i$

Def: Let $X = \{x_1, x_2, x_3, \ldots\}$ be finite or countable. $f(x_1, x_2, \ldots) = \sum_\alpha c_\alpha x^\alpha$ is a symmetric function over X if:

(a) $f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots) = f(x_1, x_2, \ldots)$ over all permutations of the index set.

(b) The monomials appearing in f have bounded degree.

The largest degree that appears is called the <u>degree</u> of f.

We regard symm. functions in a formal way, meaning that $f = g$ iff corresponding coefficients for α agree.

<u>Ex:</u> $x_1 x_2^2 + x_1 x_3^2 + x_1 x_4^2 + \ldots + x_1^2 x_2 + x_1^2 x_3 + \ldots + \ldots + x_i x_j^2 + x_j x_i^2 + \ldots$

<u>Nonexample:</u> $x_2 + x_4 + x_6 + x_8 + \ldots$

Observation: Consider $\alpha = (\alpha_1, \alpha_2, \ldots)$, the exponents of some monomial $c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots$.
Then if $f = \sum_\alpha c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots$
is symmetric, we must have
$c_\alpha = c_\beta$ where β runs through all permutations of α.
E.g. $\alpha = (1,3,1,0,0,0,\ldots)$ gives rise to $(3,1,0,1,0,\ldots)$, $(1,1,0,0,3,\ldots)$ etc. so the corresponding coeffs must be the same.

Let $\Lambda(X)$ denote the set of symmetric functions over X and let $\Lambda^m(X)$ denote the set of homogeneous symm. functions of degree m.
So $\Lambda^0(X) = ?$ (the constants)
Clearly $\Lambda(X)$ and $\Lambda_m(X)$ are vector spaces, & any $f \in \Lambda(X)$ of degree n can be uniquely written as

$$f = f_0 + f_1 + \ldots + f_m \text{ where } f_i \in \Lambda^i(X).$$

That is, $\Lambda(X)$ is the direct sum of the vector spaces $\Lambda^m(X)$.
Also, it's clear that $f \in \Lambda^m(X), g \in \Lambda^n(X)$
$\Rightarrow fg \in \Lambda^{m+n}(X)$.

So we are most interested in understanding the structure of $\Lambda^m(X)$, the homogeneous symmetric functions.

Goal: Construct three (simple) bases for $\Lambda^m(X)$. Afterwards we'll turn to a 4th basis; more interesting/complicated.

Let $X = \{x_1, x_2, \ldots\}$ countable.
We'll write $f(x)$ or f as shorthand for $f(x_1, x_2, \ldots)$ and Λ^m for $\Lambda^m(X)$.

Monomial Symmetric Functions

most obvious basis for Λ^m comes from looking at the monomials.
Suppose that $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots$ appears in $f(x) \in \Lambda^m$. Then must have $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \ldots$ in f also, where β is a permutation of α.
eg. if $x_1^0 x_2^3 x_3^4 x_4^0 x_5^1 x_6^0 x_7^0 \ldots$ in f,
must have $x_1^4 x_2^0 x_3^0 x_4^0 x_5^1 x_6^3 x_7^0 \ldots$ in f.

Idea: Rearrange elemnts of $\alpha = (\alpha_1, \alpha_2, \ldots)$ so that they are (weakly) decreasing, obtaining a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of some number m. Then define the monomial symmetric function
$$m_\lambda(x) = \sum_\alpha x^\alpha$$
where α runs through all distinct permutations of $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_r, 0, 0, \ldots)$

Elementary Symmetric Functions

already saw these for finite # variables.

$$e_k(x) = \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k} \qquad (k \geq 1)$$

$$e_0(x) = 1.$$

Note: $e_n = m_{\underbrace{1,1,\dots,1}_{n \text{ times}}}$

Observation: $\displaystyle\sum_{k \geq 0} e_k(x_1, x_2, \dots) z^k = \prod_{i \geq 1} (1 + x_i z)$

Pf: Expand the RHS.

partition of m, i.e. w/ m boxes

For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathrm{Par}(m)$, set

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_r},$$

then $e_\lambda(x) \in \Lambda^m(x)$ since degrees add to m.

Ex: $e_{211}(x_1, x_2, \dots) = (x_1x_2 + x_1x_3 + \dots)(x_1 + x_2 + \dots)^2.$

Since $\{m_\mu : \mu \in \mathrm{Par}(m)\}$ forms a basis for Λ^m, the elementary symm. functions e_λ $(\lambda \in \mathrm{Par}(m))$ are linear combinations of monomial functions m_μ.

For a matrix $A = (a_{ij})_{i,j \geq 1}$ w/ only finitely many non-zero elements, we denote by $\mathrm{row}(A) = (r_1, r_2, \dots)$ the <u>row sum vector</u>, ie. $r_i := \sum_{j \geq 1} e_{ij}$ and similarly by $\mathrm{col}(A)$ the <u>column-sum vector</u>.

<u>Prop</u>: Let $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathrm{Par}(n)$. Then

$$e_\lambda = \sum_{\mu \in \mathrm{Par}(n)} M_{\lambda\mu} m_\mu$$

where $M_{\lambda\mu}$ = # of 0-1 matrices $A = (a_{ij})_{i,j \geq 1}$ w/ $\mathrm{row}(A) = \lambda$, $\mathrm{col}(A) = \mu$.