Math 155 – Lecture 1

★ Hand out survey and syllabus

About the course:
Algebraic combinatorics is about using algebraic techniques to study combinatorial objects or vice-versa.

In particular, in this class we will see interactions between combinatorics & representation theory.

Representation theory is the study of how group can act on a vector space. Philosophy is that to really understand a gp, we need to understand how it can act...

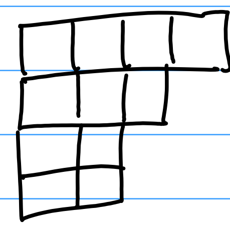
One of our main topics will be comb. & rep theory of symmetric group S_n. It turns out that the rep theory can be described in terms of partitions, Young diagrams, and standard tableaux

Def: A partition is a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$.
$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$

We say λ is a partition of n if $\sum \lambda_i = n$

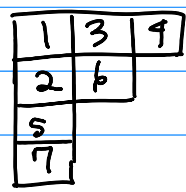
A Young diagram is a way to visualize a partition: we draw an arrangement of boxes w/ λ_i boxes in row i.

eg. $(4,3,2,2) \longleftrightarrow$



A standard tableau of shape λ is a filling of the Young diagram of λ w/ distinct pos. integers from 1 to n s.t. the rows & columns are increasing from L to R and top to bottom

eg

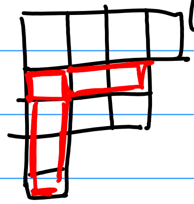


Some motivation:

It will turn out that the irreducible reps of S_n are in bijection with partitions λ of n (so we can call each one V_λ), and $\dim V_\lambda$ = # of standard tableaux of shape λ. In fact we can construct a basis for V_λ using the std tableaux.

There is lots of amazing combinatorics here. Here's a result that we will see later:

Def: The <u>hook</u> of a box b in a Young diagram D is the set of all boxes in D to the right or below b, along w/ b.



The <u>hook length</u> h_b of b is # of boxes in hook of b.

Theorem: Let λ be a partition of n. The # of std tableaux of shape λ is

$$\frac{n!}{\prod_{b \in \lambda} h_b}$$

So this $= \dim V_\lambda$

Ex: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$ $\quad$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$ $\quad$ vs $\quad \dfrac{3!}{1 \cdot 1 \cdot 3} = 2$

hook lengths are: $\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}$

Next: Rep theory of finite gps (2 or 3 lectures?)

Q: How many people have seen rep theory?

Def: A <u>representation</u> of a finite group G on a finite-dimensional complex vector space V is a homomorphism $\rho: G \to GL(V)$; $GL(V) =$ gp of automorphisms of V. If we choose a basis $\{e_1, \ldots, e_n\}$ for V, we can identify $GL(V)$ with $GL_n(\mathbb{C})$, the group of

$n \times n$ invertible matrices.

Note that the map gives V the structure of a G-module: we can define an action of G on V by $g.v := \rho(g)v$.

When the map ρ is understood we sometimes just call V a rep. of G and use the notation gv or $g.v$ instead of $\rho(g)v$.

The <u>degree</u> of ρ (sometimes called <u>dimension</u>) is $\dim V$.

<u>Ex:</u> Let G be any group + $V = \mathbb{C}^1$

Define $\rho: G \to GL(V)$ by $\rho(g) = \text{identity}$.

equiv $\rho: G \to GL_1(\mathbb{C})$ by $\rho(g) = I_1 = 1$

Then for any $v \in V$, $g.v = v$.

This is called the <u>trivial</u> representation.

<u>Ex:</u> Let G be any group. Suppose X is a finite set and G acts on the left on X, i.e. we have a map $G \times X \to X$ denoted $(g,x) \mapsto g.x$ s.t.

$(gh).x = g.(h.x) \quad \forall\ g,h \in G$ and $x \in X$,

and $e.x = x \quad \forall\ x \in X$.

$\therefore$ we have a homomorphism $G \to \mathrm{Aut}(X)$ from G to the permutation group of X.

(each $g \in G$ permutes elemts of X)

($g.x_1 = g.x_2 \Rightarrow x_1 = x_2$)

So we have a permutation representation of G:
let V be the v.s. w/ basis $\{e_x : x \in X\}$
and let G act on V by

$$g.\, e_x = e_{gx} \quad \text{+ extending linearly -}$$

i.e. $$g.\, \Sigma a_x e_x = \Sigma a_x e_{gx}.$$

When we let $X = G$ + let G act on itself, the representation we get in this way is called the regular representation.
Note that this rep has dimension $|G|$.

One of our main examples of G will be symm gp.
Let S_n be the symmetric group on n letters. Notation for permutations:

216534 or

(12)(3645)

Multiplication: $(12)(34) \cdot (1234) \cdot (142)(3)$ means $(12)(3)(4)$.
(read R to L)

The permutation $(123)(4)(5)$ is often just written as (123) for short.

Trivial rep of S_n $\Leftrightarrow$ Young diagram $\overbrace{\boxed{}\boxed{}\cdots\boxed{}}^{n \text{ boxes}}$ Check hook length formula!

The transpositions are those elements of S_n of the form (ij)
The adjacent transpositions in S_n are:

$$\underset{s_1}{(12)},\ \underset{s_2}{(23)},\ \underset{\cdots}{\cdots},\ \underset{s_{n-1}}{(n-1\ n)}.$$

Fact: S_n is generated by adjacent transpositions.

Def: Let $\pi \in S_n$ and write π as a product of adjacent transpositions $s_{i_1} \dots s_{i_n}$. Define $\mathrm{sgn}(\pi) = (-1)^n$.

Fact: This is a homomorphism $S_n \to \{\pm 1\}$.

This gives rise to a 1-dimensional rep of S_n. We let V be a vect. space w/ basis $\{v\}$ and let $\rho: S_n \to GL(V) = GL_1(\mathbb{C}) = \mathbb{C}^*$ be
$$\rho(\pi) = \mathrm{sgn}(\pi).$$
This is called the sign rep. of S_n and $\longleftrightarrow$ Young diagram  $\swarrow$ n boxes. Note: hook formula gives 1.

Def: A map φ between two reps V and W of G is a vect. space map $\varphi: V \to W$ s.t.
$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"'] & W \end{tikzcd}$$
commutes $\forall g \in G$.

Also called a G-linear map.

Exercise: $\mathrm{Ker}\,\varphi$, $\mathrm{Im}\,\varphi$, and $\mathrm{Coker}\,\varphi$ are also G-modules.

Def: A subrepresentation of a rep V is a vector subspace W of V which is invariant under G. (i.e. $\forall w \in W$, and $\forall g \in G$, $g.w \in W$)

Def: A rep. V is called irreducible if there is no proper nonzero invariant subspace W of V.

Ways to get new reps from reps we already have:

① If V and W are representations, the <u>direct sum</u> $V \oplus W$ is also a rep:
we let $g.(v,w) := (g.v,\ g.w)$

② The tensor product:
Let V and W be vector spaces w/ bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$.
Then $V \otimes W$ is the vector space w/ basis $\{v_i \otimes w_j \mid 1 \le i \le n,\ 1 \le j \le m\}$.
The construction of tensor products is

comm. $\quad V \otimes W \cong W \otimes V$ via $v \otimes w \mapsto w \otimes v$

distrib. $\quad (V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$

assoc. $\quad (U \otimes V) \otimes W \cong U \otimes (V \otimes W) \cong U \otimes V \otimes W$

In particular there are <u>tensor powers</u> $V^{\otimes n} = V \otimes \dots \otimes V$.

If V and W are reps then so is $V \otimes W$:
$g.(v \otimes w) := (g.v \otimes g.w)$.

<u>Goal</u>: We would like to attempt to classify the reps of a finite group.
Would be nice to simplify life by restricting our search somewhat — we know that we can build new reps of G from old ones by taking direct sum — so would like to focus on reps which are "atomic" w/ respect to this operation — i.e. cannot be expressed as direct sum of others. Such a rep is called <u>indecomposable</u>.

It turns out that a rep is indecomposable iff it is irreducible (contains no proper subreps) + every rep is a direct sum of irreducibles. This follows from the next prop:

(Maschke's Theorem)

Prop: If W is a subrepresentation of a representation V of finite gp G then there is a complementary G-invariant subspace W' of V s.t. $V = W \oplus W'$.

Proof: Choose an arbitrary subspace U complementary to W in V. Problem is it is probably not G-invariant. To fix this, let $\pi_0 : V \to W$ be the projection given by the direct sum decomposition $V = W \oplus U$ and average map π_0 over G:

let $$\pi(v) = \sum_{g \in G} g(\pi_0(g^{-1}v)).$$

This is a G-linear map from V onto W. When restricted to W, the map is mult by $|G|$. By exercise, $\ker \pi$ is a subspace of V invariant under G. It is complementary to W so we are done.

Question: Why doesn't this work for infinite groups?

Cor: Any representation (of a finite group G) is a direct sum of irreducible representations (irreps).

This property is called <u>complete reducibility</u> or semisimplicity.

To what extent is the decomposition of an arbitrary representation into a direct sum of irreducibles unique?

<u>Schur's Lemma</u>: If V and W are irreps of G and $\varphi: V \to W$ is a G-module homomorphism then
(1) either φ is an isomorphism or $\varphi = 0$
(2) If $V = W$ then $\varphi = \lambda I$ for some $\lambda \in \mathbb{C}$, I the identity.

<u>Proof</u>: $\ker \varphi$ and $\operatorname{im} \varphi$ are subrepresentations of V.
$\therefore$ Since V is irreducible, $\ker \varphi = \phi$ or $\ker \varphi = V$. and $\operatorname{im} \varphi = \phi$ or $\operatorname{im} \varphi = W$.
If $\ker \varphi = \phi$ then $\operatorname{im} \varphi$ must be $W \Rightarrow$ φ an isomorphism. If $\ker \varphi = V$ then $\varphi = 0$. $\therefore$ (1).
If $V = W$ (and $\varphi \neq 0$) then φ is an isomorphism. Since $\mathbb{C}$ alg closed, φ has an eigenvalue λ. (can choose basis for V & write φ as matrix)
Then $\varphi - \lambda I$ has a nonzero kernel. Since it is a G-module h'ism, its kernel is V $\Rightarrow$ $\varphi = \lambda I$.

<u>Prop</u>: For any rep V of a finite group G, there is a decomposition
$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$ where the V_i are distinct irreps. The decomp. of V into a direct sum of the k factors

is unique, as are the V_i that occur & the multiplicities.

Proof: Suppose we have another decomp.
$V = W_1^{\oplus b_1} \oplus \dots \oplus W_\ell^{\oplus b_\ell}$.
Clearly $\varphi = id$ is a map of representations.
By Schur's Lemma, φ must map the factor $V_i^{\oplus a_i}$ into that factor $W_j^{\oplus b_j}$ for which $W_j \cong V_i$. But then $a_i = b_j$ and the result follows.

Since each rep decomposes uniquely into a direct sum of irreducibles, to classify reps of G just need to classify the irreps.
What will be helpful is that it turns out that each G has only finitely many irreps.

An easy case: abelian groups.

Prop: Every irrep of an abelian group G is 1-dim.
Proof: Let V be an irrep of G.
Choose $g \in G$. This defines a G-linear map $\rho: V \to V$ defined by $v \mapsto g.v$.
It is G-linear because $h.(g.v) = hg.v = gh.v = g.(h.v) \Rightarrow h.\rho(v) = \rho(h.v)$.
So by Schur's Lemma, each $g \in G$ acts by a scalar multiple of the identity.

$\therefore$ for any $v \in V$ and $g \in G$, $g.v = \lambda v$ for a scalar λ $\Rightarrow$ $\{v\}$ spans a subrep of V $\Rightarrow$ since V is irreducible, $V = \langle v \rangle$. (span of v).

So irreducible reps of G are just homomorphisms $\rho : G \to \mathbb{C}^*$

__Next:__ Consider $G = S_3$.
Already we know we will have the trivial one-dim rep and the sign rep. Also, since G is a permutation group, we can let $\{e_1, e_2, e_3\}$ be a basis for $\mathbb{C}^3$ & let G act by $g.e_i = e_{g(i)}$.

In other words, the map $\rho : G \to GL(\mathbb{C}^3)$ is given by:

$$\mathrm{id} \mapsto \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$$(12) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{etc.}$$

However, this rep is not irreducible — why? What is the invariant subspace?