MATHEMATICS 154, SPRING 2009
MATHEMATICAL PROBABILITY
Check List for the Final Exam

1 Definitions and simple theorems

These are appropriate subject matter for multiple-choice or true-false items.

1. Countability - finite Cartesian products, countable unions and arbitrary intersections of countable sets are countable. The collection of all subsets of a countably infinite subset is uncountable.

2. Definition of a σ-field - closure under complement and countable union implies closure under difference and intersection. Example of two σ-fields whose union is not a σ-field.

3. Inclusion-exclusion formula for two events, and related inequalities.

4. Probabilities for increasing and decreasing infinite sequences of events.

5. Definition of conditional probability.

6. Definition of independent events; example that three events can be pairwise independent but not independent.

7. Probability of k successes in n independent identical experiments.

8. Geometric distribution - probability that the first success in a sequence of identical independent experiments occurs on the kth attempt.

9. Definition of random variables - discrete, continuous, and those that are neither.

10. General properties of distribution functions for random variables.

11. Definition of independence for discrete random variables.

12. Probability mass functions for binomial, Poisson, geometric and negative binomial distributions.

13. Definition of expectation and variance for discrete random variables; comment on the role of absolute convergence here.

14. Definition of "uncorrelated" for two random variables; example of two random variables that are uncorrelated but not independent.

15. Statement and proof of the "law of the unconscious statistician" for one discrete random variable.

16. Expectation of the sum of random variables.

17. Variance of the sum of independent random variables and of a multiple of a random variable.

18. Definition of covariance and correlation for two random variables.

19. Basic properties of random walks - homogeneity in time and space. Markov property.

20. Random walks - formula for number of paths in n steps from level a to level b.

21. Reflection principle for random walks - use to obtain a formula for the number of paths from level a to level b that do not cross level 0.

22. Ballot theorem, including the equivalent case of the "hitting time theorem:" the fraction of paths from level 0 to level b in N steps that reach level b for the first time at the Nth step.

23. Key lemma for arc sine laws. For a symmetric random walk ($p = \frac{1}{2}$) that starts in level 0, the following are all equal:

    - u_{2m} = the probability of being in level 0 after $2m$ steps.
    - u_{2m} = the probability of never revisiting level 0 during $2m$ steps.
    - u_{2m} = the probability of revisiting level 0 exactly once during $2m$ steps.
    - u_{2m} = the probability of never visiting a negative level during $2m$ steps.
    - u_{2m} = the probability of never visiting a positive level during $2m$ steps.

24. Arc sine law for last return (mass function): after $2n$ steps of a symmetric random walk, the probability that the last visit to level 0 was at step $2k$ is exactly $u_{2k}u_{2n-2k}$ and approximately

$$\frac{1}{\pi\sqrt{k(n-k)}}$$

25. Arc sine law for last return (distribution function): after $2n$ steps of a symmetric random walk, the probability that the last visit to level 0 was before step $2xn$ is approximately $\frac{2}{\pi}\arcsin\sqrt{x}$.

26. General properties of density functions for continuous random variables.

27. In terms of the density function $f_X(x)$, integrals for expectations $\mathbb{E}(X)$ and $\mathbb{E}(X^2)$.

28. "Law of the unconscious statistician" for the expectation of a function $Y = g(X)$ of a continuous random variable.

29. Exponential distribution and its expectation.

30. Density function for normal distribution $N(0, 1)$ - the variance is 1.

31. Density function for normal distribution $N(\mu, \sigma)$ - the mean is μ and the variance is σ.

32. For a pair of random variables, express the distribution function $F_{X,Y}(x, y)$ in terms of the density function $f_{X,Y}(x, y)$ and vice versa.

33. "Law of the unconscious statistician" for calculating the expectation of a random variable $Z = g(X, Y)$.

34. Definition of marginal density and conditional expectation.

35. Definition of independence for two random variables; what this means in terms of the density function.

36. Density function for the sum of two independent random variables X and Y as a convolution of the density functions for X and Y.

37. Bertrand's paradox: state the condition for an equilateral triangle based on a random chord to fit inside a circle. Describe three reasonable but different interpretations of "random chord," and for each of them determine the probability that the triangle fits inside the circle.

38. Crofton's method: the general strategy for turning a geometrical probability problem into a differential equation.

39. Generating functions: definition of $G(s)$ in terms of the probability mass function. Determining a value of the probability mass function by evaluating an appropriate derivative of $G(s)$. Significance of $G(1)$.

40. Generating function for the sum of two or more independent random variables or for the sum of n independent copies of a random variable.

41. Statement of the "weak law of large numbers."

42. Statement of the central limit theorem, and the recipe that it implies for generating random numbers with a normal distribution.

43. Statement of Markov's inequality and Chebyshov's inequality, and use to obtain crude bounds on how fast the probability must fall to zero for large values of a random variable with zero mean.

2 Applications

These can generate a wide variety of problems, some of which are simple enough to cast as multiple-choice items or short-answer questions.

1. Equations and inequalities that follow by using induction and the inclusion-exclusion formula.
2. Calculating probabilities by using the inclusion-exclusion formula for two or three events.
3. Probabilities for poker hands.
4. Probabilities for bridge - distribution of suits, or distribution of missing cards in a suit between two opposing hands.
5. Conditional probability problems involving two events (the "bearded man" problem).
6. Generalizations of the Monty Hall problem.
7. Problems based on "Bernoulli trials" - a certain number of successes in n independent identical experiments.
8. Analysis of Simpson's paradox in terms of conditional probability.
9. Problems based on geometric and negative binomial distributions - what is the probability that it takes exactly n trials to get k successes?
10. "Problem of the points" - what is the probability that m successes occur before n failures?
11. Problems involving the "probabilistic method" and the pigeonhole principle.
12. Problems based on the Poisson distribution: use the Poisson distribution to find an approximate solution to a problem that involves a binomial distribution with large n and small p.
13. Random walk problems, solved by using conditional probability to write down a linear difference equation and then solving the difference equation with appropriate boundary conditions.
14. Given the density or distribution function for a random variable X, determine its expectation and variance.
15. Given the density or distribution function for a random variable X and a new random variable $Y = g(X)$, determine the density function and expectation of Y.

16. Given a random variable X with a uniform distribution on [0,1], invent a random variable $Y = g(X)$ that has some specified distribution function or density function.

17. Given a joint density function $f_{X,Y}(x, y)$, determine whether or not X and Y are independent.

18. Given a joint density function $f_{X,Y}(x, y)$, calculate the conditional density function $f_{Y|X}(y, x)$.

19. Given a joint density function $f_{X,Y}(x, y)$ and new random variables U and V defined by formulas $u(x, y)$ and $v(x, y)$, determine the density function $f_{U,V}(u, v)$.

20. Given the density functions for two independent random variables, calculate the density function for their sum.

21. Given a specification of a way to select the "random chord" for Bertrand's paradox, calculate the joint density function for the polar coordinates of the midpoint of the chord.

22. Solve geometric probability problems involving two points by dividing the region of interest in half and conditioning on whether the two points lie in the same half or in different halves.

23. Solve geometric probability problems where one of the points of interest lies at a randomly chosen point on a line or circular arc, by conditioning on the position of that point.

24. Solve geometric probability problems by Crofton's method, considering first the simpler case where a point of interest lies on the boundary, then setting up a differential equation.

25. Write down the generating function for a discrete random variable that can take a small number of integer values and for the sum of several such variables.

26. Given the generating function for a random variable X, determine the expectation and variance of X.

27. Given a generating function for the sum S of k independent copies of a random variable, expand in a power series to find the probability that $S = n$. For example, what is the probability to throw a total of n with k fair dice?

28. Use generating functions to solve problems involving the sum of random variables with the geometric distribution.

29. Given a random variable with a parameter that is itself a random variable, find the generating function that results from summing or integrating over the random parameter.

30. Given a random variable X (discrete or continuous), determine what the law of large numbers has to say about the average of many such random variables.

31. "Munchkin problems" (the hypergeometric distribution): given n objects selected from N of which b are special, find the probability that k of the n are special.

32. Use Waring's theorem to relate probabilities of various events that involve the occurrence of subsets of n events.