MATHEMATICS 154, SPRING 2009
PROBABILITY THEORY
Proof list for final exam

Last revised: May 9, 2009

The final exam will include five of these proofs.

1. State the axioms for a probability function, and use them to prove the inclusion-exclusion rule for two events A and B. Then extend the rule to the case of three events A, B, and C.

2. Starting from the axioms for a probability function and the definition of the distribution function $F_X(x)$ for random variable X, prove that $F_X(x)$ is nondecreasing and that

$$\mathbb{P}(X = x) = F_X(x) - \lim_{h \to 0} F_X(x - h),$$

where the limit is one-sided, for positive values of h only.

3. You start with k chips and play a fair game ($p = \frac{1}{2}$) until you either have 0 chips or N chips. Prove that the expected number of plays is $D_k = k(N - k)$.

4. Prove that for a random walk of $2n$ steps that starts in level 0, the number of "nonnegative" walks is equal to the number of "return to zero" walks.

5. Let X be a continuous random variable with density function $f_X(x)$, for which $\mathbb{P}(X < 0) = 0$. Let $Y = g(X)$, where g is a non-negative function. Starting from the tail-sum formula for the expectation of Y, prove that

$$\mathbb{E}(Y) = \int_0^\infty f(x)g(x)dx.$$

If you consider a double integral as part of your proof, draw a diagram to show the region of integration.

6. Prove that if X and Y are independent random variables, each having the exponential distribution function with parameter λ, then their sum has the $\Gamma(2)$ distribution.

7. Suppose that a random chord of a circle is chosen by selecting two points uniformly and independently on the circle and joining them to get the chord. An equilateral triangle is then constructed with this chord as one side. Prove that the probability that this triangle fits inside the circle is $\frac{2}{3}$.

8. Using the change of variables formula involving the Jacobian, derive the convolution formula

$$\frac{1}{2}\int_{-\infty}^{\infty} f_{X,Y}(\frac{z+v}{2}, \frac{z-v}{2})dv.$$

(Hint: define $Z = X + Y$ and $V = X - Y$.)

Use it to prove that if X and Y are independent random variables, each having the $N[0, 1]$ distribution function, then their sum has the $N[0, 2]$ distribution.

9. Prove that if X and Y are discrete random variables that take on only non-negative integer values, then the generating function for their sum $Z = X + Y$ is the product of the generating functions for X and Y. Then prove that if $U = X_1 + X_2 + \cdots + X_n$, where all the X_i are independent copies of X, then $G_U(s) = G_X(s)^n$.

10. First, derive formulas for the expectation and variance of a random variable X whose generating function is $G_X(s)$. Second, find the generating function for a Poisson random variable with parameter λ. Finally, combine these results to prove that both the expectation and variance of a Poisson random variable are equal to λ.

11. Suppose that X and Y are chosen uniformly and independently in $[0, z]$. Use Crofton's method to compute the expectation of $|Y - X|$. (Be careful to justify any approximations that you make.)

12. Assume that random variable X has expectation 0 and variance σ^2. Use properties of its generating function to prove the central limit theorem.

13. Prove Chebyshov's inequality and use it to prove the weak law of large numbers.