

Problem Set 4

February 29, 2016

- (1) Let \mathbf{R} be the set of real numbers, and let \mathcal{F} be a collection of subsets of \mathbf{R} . Suppose that \mathcal{F} is a σ -algebra and that \mathcal{F} contains the interval (a, b) , whenever a and b are rational numbers satisfying $a < b$. Show that \mathcal{F} contains every open subset of \mathbf{R} .
- (2) Let \mathbf{R} be a set of real numbers, and let \mathcal{F} denote the collection of subsets $A \subseteq \mathbf{R}$ for which either A or its complement $A^c = \mathbf{R} - A$ is countable.
 - (a) Show that \mathcal{F} is a σ -algebra of subsets of \mathbf{R} .
 - (b) Show that the function $P : \mathcal{F} \rightarrow \mathbf{R}_{\geq 0}$ given by

$$P(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{otherwise.} \end{cases}$$

makes $(\mathbf{R}, \mathcal{F}, P)$ into a probability space.

For the remaining problems, we let (Ω, \mathcal{F}, P) denote a probability space.

- (3) Let $X, Y : \Omega \rightarrow \mathbf{R}$ be random variables. Show that the set $\{\omega \in \Omega : X(\omega) = Y(\omega)\}$ is an event.
- (4) Let $X, Y : \Omega \rightarrow \mathbf{R}$ be random variables. We say that X and Y are *equal almost everywhere* (or *equal almost surely*) if $P(\{\omega \in \Omega : X(\omega) = Y(\omega)\}) = 1$. Show that this is an equivalence relation on the set of random variables.
- (5) Suppose that the set Ω is countable. Show that there exists a set Ω' , a function $P' : \Omega' \rightarrow \mathbf{R}_{\geq 0}$ satisfying $\sum_{\omega' \in \Omega'} P'(\omega') = 1$, and a surjective function $f : \Omega \rightarrow \Omega'$ with the following property:
 - (*) A subset $A \subseteq \Omega$ belongs to \mathcal{F} if and only if $A = f^{-1}(A')$ for some subset $A' \subseteq \Omega'$. In this case, we have $P(A) = \sum_{\omega' \in A'} P'(\omega')$.