Problem Set 3

February 17, 2016

(1) Fix a real number $p \in (0, 1)$. Suppose that we repeatedly flip a biased coin (heads with probability p, tails with probability $(1-p)$) until it comes up heads. What is the expected number of coinflips?

(2) Let $\Omega = \{0, 1, 2, \ldots\}$ be the set of nonnegative integers, let $P : \Omega \to \mathbf{R}$ be the Poisson distribution with mean $\lambda \geq 0$, and let $X : \Omega \to \mathbf{R}$ be the function $X(n) = n$. Compute the third moment $\mathbf{E}[X^3]$.

(3) Let Ω be a countable set and suppose we are given a sequence $P_0, P_1, P_2, \ldots,$ where each $P_n : \Omega \to \mathbf{R}_{\geq 0}$ is a probability distribution on Ω. Assume that for each $\omega \in \Omega$, the sequence of real numbers $\{P_n(\omega)\}_{n \geq 0}$ converges to some real number $P(\omega)$. Show that $\sum_{\omega \in \Omega} P(\omega)$ converges, but give an example where it does not converge to 1.

(4) In the situation of (3), suppose that $\sum_{\omega \in \Omega} P(\omega) = 1$ (so that P is also a probability distribution on Ω). Let $X : \Omega \to \mathbf{R}_{\geq 0}$ be a nonnegative function. Suppose that, for each $n \geq 0$, the function X has finite expectation $\mathbf{E}_n[X]$ when regarded as a random variable on (Ω, P_n). Show that if the sequence of real numbers $\{\mathbf{E}_n[X]\}_{n \geq 0}$ is bounded, then X has finite expectation when regarded as a random variable on (Ω, P).

(5) Give an example of a function $X : \Omega \to \mathbf{R}_{\geq 0}$ satisfying the hypotheses of (4) where the sequence $\{\mathbf{E}_n[X]\}_{n \geq 0}$ converges, but does not converge to $\mathbf{E}[X] = \sum_{\omega \in \Omega} P(\omega) X(\omega)$.