Problem Set 2

February 9, 2016

Let (Ω, P) be a finite probability space.

(1) Let $X, Y : \Omega \to \mathbb{R}$ be random variables and let $f, g : \mathbb{R} \to \mathbb{R}$ be functions. Show that if X and Y are independent, then $f(X)$ and $g(Y)$ are independent.

(2) Assume that for each outcome $\omega \in \Omega$, the probability $P(\omega)$ is positive. Show that if a random variable $X : \Omega \to \mathbb{R}$ is independent of itself, then X is constant.

(3) Let $\Omega = \{1, 2, 3, 4, 5, 6\}^2$ let P be the uniform distribution (so that (Ω, P) models a pair of fair dice). Define random variables $X, Y : \Omega \to \mathbb{R}$ by the formulae $X(i, j) = \min\{i, j\}$, $Y(i, j) = \max\{i, j\}$. Compute the quantities $\mathbf{E}(X)$, $\mathbf{E}(Y)$, $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, and $\mathrm{Cov}(X, Y)$.

(4) Let X and Y be random variables with standard deviations σ_X and σ_Y. Show that $|\mathrm{Cov}(X, Y)| \leq \sigma_X \sigma_Y$.

(5) Let $X : \Omega \to \mathbb{R}$ be a random variable, let t be a real number, and define $X' : \Omega \to \mathbb{R}$ by the formula $X'(\omega) = \max\{X(\omega), t\}$. Show that $\mathrm{Var}(X') \leq \mathrm{Var}(X)$.