Problem Set 1

February 1, 2016

Let (Ω, P) be a finite probability space.

(1) Let $X : \Omega \to \mathbb{R}$ be a random variable. Assume that $X \geq 0$, and let $A = \{\omega \in \Omega : X(\omega) > 0\}$ be the event that X is positive. Show that if $A \neq \emptyset$, then $P(A) \geq \frac{\mathbf{E}(X)^2}{\mathbf{E}(X^2)}$.

(2) Let $X : \Omega \to \mathbb{R}$ be a random variable, let t be a real number, and let $A = \{\omega \in \Omega : X(\omega) \geq t\}$ be the event that $X \geq t$. Show that $P(A) \leq \frac{\mathbf{E}(e^X)}{e^t}$.

(3) Let (Ω, P) be a finite probability space, let $A_1, \ldots, A_n \subseteq \Omega$ be events, and let $A = \bigcup_{1 \leq i \leq n} A_i$. Show that

$$\sum_{1 \leq i \leq n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j) \leq P(A) \leq \sum_{1 \leq i \leq n} P(A_i).$$

(4) Let $X : \Omega \to \mathbb{R}$ be a random variable. Show that the function $t \mapsto \mathbf{E}((X - t)^2)$ attains its minimum when $t = \mathbf{E}(X)$.

(5) Show that if every outcome $\omega \in \Omega$ has positive probability and $X, Y : \Omega \to \mathbb{R}$ are nonconstant random variables satisfying $X = f(Y)$ for some function $f : \mathbb{R} \to \mathbb{R}$, then X and Y cannot be independent. Give an example to show that in this situation, it is possible for X and Y to be uncorrelated.