

## Math153, Fall 2004, Problem Set 1 Solutions

Due before 4PM on Tuesday October 12th, 2004, at PED office at 1 Brattle Square. No late homework will be accepted. For full credit, please be sure to show and explain your work.

1. (10pts) Derive from first principles the differential equation associated to the difference equation

$$x_{t+1} = (1 + a)x_t.$$

*Show all your work.*

Solution:

$$\begin{aligned}x_{t+1} &= (1 + a)x_t \\x_{t+h} &= (1 + c)x_t\end{aligned}$$

for small  $h$ , and

$$(1 + c)^{1/h} = 1 + a$$

For small  $h$ , we use Taylor expansion to estimate

$$(1 + a)^h = 1 + \text{Log}(1 + a)h + O(h^2)$$

Hence

$$\begin{aligned}x(t + h) &= (1 + \text{Log}(1 + a)h)x(t), & \frac{x(t + h) - x(t)}{h} &= (\text{Log}(1 + a))x(t) \\ \dot{x}(t) &= (\text{Log}(1 + a))x(t)\end{aligned}$$

2. Analyze the dynamical properties of the non-linear difference equation

$$x_{t+1} = x_t e^{r(1-x_t)}$$

- (a) (2pts) Decide what the range of  $x_t$  should be.

Solution:  $x_t \in [0, \infty)$ .

- (b) (3pts) Decide what the range of  $r$  should be in order to preserve the property that  $x_t$  increases at low densities and decreases at high densities.

Solution: Let

$$f(x) = x e^{r(1-x)}, \quad \frac{df}{dx} = e^{r(1-x)}(1 - rx)$$

$r \in (0, \infty)$ .  $r$  can not be negative, otherwise, for  $x_t$  is large,  $1 - rx_t > 0$ , hence

$$\frac{df}{dx} > 0$$

then  $x_t$  would still increase at high densities. If  $r = 0$ , then  $x_t$  is constant.

Another way to see this is that we wanted the function  $f(x)$  to be one-humped, i.e.

$$\frac{df}{dx} = 0$$

exactly once. This can only occur if  $r > 0$ .

- (c) (3pts) Find the fixed point of this difference equation.

Solution:

$$x_t = x_t e^{r(1-x_t)}, \quad x_t = 1$$

So  $p = 1$  is the fixed point of the system.

- (d) (5pts) Find the range of  $r$  so that  $p = 1$  is asymptotically stable, in the sense that for all  $x_t$  close to  $p = 1$ ,  $x_{t+1}$  is even closer to  $p$ , i.e.

$$|x_{t+1} - p| < |x_t - p|$$

Solution: Let

$$f(x) = x e^{r(1-x)}$$

By the mean value theorem, we know that

$$f(x) - p = f(x) - f(p) = (x - p) \frac{df}{dx}(c)$$

for some  $c \in (x, p)$ . If  $|\frac{df}{dx}(p)| < 1$ , then  $|\frac{df}{dx}(c)| < 1$  if  $x$ , and hence  $c$ , is sufficiently close to  $p$ . Therefore,

$$|f(x) - p| < |x - p|$$

, i.e.  $x_{t+1}$  is closer to  $p$  than  $x_t$  for  $x_t$  sufficiently close to  $p$ .

$$\left| \frac{df}{dx}(p) \right| = |1 - r| < 1$$

implies that

$$r \in (0, 2)$$

- (e) (10pts) Let  $r = 2.4$ , use Mathematic/Matlab/Maple, to plot  $y = x$ ,  $y = f(x)$ , and  $y = f(f(x))$ , where

$$f(x) = x e^{r(1-x)}.$$

Choose the domain of  $x$  carefully so that you can see the bifurcation phenomenon, and label all the fixed/periodic points.

Solution: See Figure 1.

3. (15pts) Solve the difference equation

$$x_{n+1} = 2x_n + x_{n-1} + 3, \quad x_0 = -\frac{3}{2} \quad x_1 = -\frac{1}{2}$$

The sequence is  $-1.5, -0.5, 0.5, 3.5, 10.5, \dots$

Solution: The complete solution  $x_n$  is the sum of the particular solution and homogeneous solution:

$$x_n = x_n^p + x_n^h$$

The inhomogenous (particular) solution to

$$x_{n+1} = 2x_n + x_{n-1} + 3$$

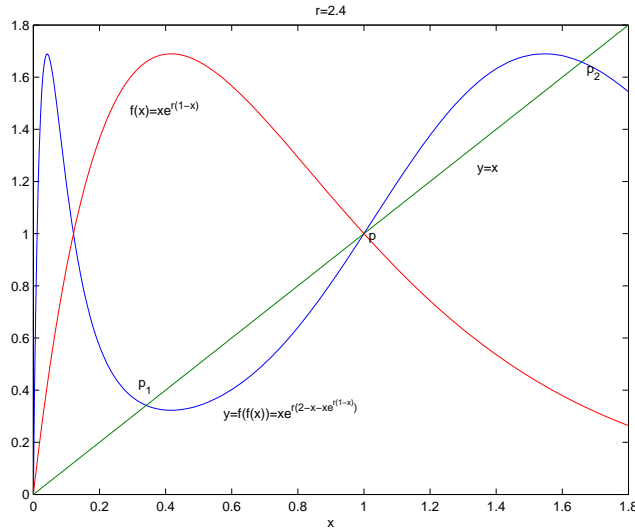


Figure 1:

is  $x_n^p$ .

$$c = 2c + c + 3, \quad x_n^p = c = -\frac{3}{2}$$

The characteristic equation

$$x^2 - 2x - 1 = 0$$

has roots

$$r_1 = 1 + \sqrt{2}, \quad r_2 = 1 - \sqrt{2}$$

The homogeneous solutions to

$$x_{n+1} = 2x_n + x_{n-1}$$

is

$$x_n^h = c_1 r_1^n + c_2 r_2^n$$

Since

$$x_0^h = x_0 - x_0^p = 0, \quad x_1^h = x_1 - x_1^h = 1$$

$c_1, c_2$  are determined by initial conditions to be

$$c_1 = \frac{\sqrt{2}}{4}, \quad c_2 = -\frac{\sqrt{2}}{4}$$

The complete solution is therefore

$$x_n = \frac{\sqrt{2}}{4}(1 + \sqrt{2})^n - \frac{\sqrt{2}}{4}(1 - \sqrt{2})^n - \frac{3}{2}$$

4. Consider the following system of ODE

$$\begin{aligned} \dot{x} &= -x + xy \\ \dot{y} &= y - xy \end{aligned}$$

where  $x$  is the amount of predator,  $y$  the amount of prey.

- (a) (8pts) Find all the critical points of this system, and decide the nature of the critical points.

Solution: Critical Point:  $(0, 0)$  and  $(1, 1)$ .  $(0, 0)$  is a saddle, and  $(1, 1)$  is a center.

- (b) (3pts) Suppose  $x + y = 1$  always. Find  $\phi$ , the average fitness of the entire population, for the frequency dependent selection system

$$\begin{aligned}\dot{x} &= -x + xy - \phi x \\ \dot{y} &= y - xy - \phi y\end{aligned}$$

Solution: WLOG, suppose  $x + y = 1$  always. We have

$$\phi = y - x$$

- (c) (3pts) Using the  $\phi$  obtained above to analyze the behavior of this system.

Solution: We have

$$\begin{aligned}\dot{x} &= x(x - 1) \\ \dot{y} &= y(1 - y)\end{aligned}$$

The system is completely decoupled.

The equation  $\dot{x} = x(x - 1)$  has two equilibrium solutions:  $x = 0$  and  $x = 1$ . The Jacobian is  $J = 2x - 1$ . So 1 is an unstable equilibrium (source node), while 0 is a stable equilibrium (sink node).

Similarly,  $\dot{y} = y(1 - y)$  has an unstable equilibrium at  $y = 0$  and a stable equilibrium at  $y = 1$ .

Selection favors the population to consist of prey entirely. So as  $t \rightarrow \infty$ , we have  $x = 0$  and  $y = 1$ .

- (d) (8pts) Suppose initially  $x = 1/2$ , using the same  $\phi$ , solve for  $x$  explicitly.

Solution:

$$\dot{x} = x(x - 1)$$

Using separation of variable and partial fractions, we have

$$\left( \frac{1}{x-1} - \frac{1}{x} \right) dx = dt, \quad \frac{x-1}{x} = ce^t$$

$$x = \frac{1}{1 + ce^t}$$

$x(0) = 1/2$  implies  $c = 1$ . So

$$x(t) = \frac{1}{1 + e^t}$$

5. (20pts) **Protease Inhibitor:** Protease inhibitor of HIV prevent infected cells  $y$  from producing infectious virus particles  $v$ . Free virus particles, which have been produced before therapy starts, will for a short while continue to infect new cells. The infected cells, however, will produce non-infectious virus particles  $x$ . We have the following system of equations to describe the dynamics:

$$\begin{aligned}\dot{y} &= \beta xv - ay \\ \dot{v} &= -uv \\ \dot{w} &= ky - uw\end{aligned}$$

We further assume that  $x$ , the amount of uninfected cells, remain constant for the time-scale of investigation at

$$x = x^* = \frac{au}{\beta k}$$

Also assume that

$$y(0) = y^* = \frac{du}{\beta k}, \quad v(0) = v^* = \frac{d}{\beta}, \quad w(0) = 0$$

- (a) Solve this linear system of equations.

Solution: The matrix associated to this linear system is

$$A = \begin{pmatrix} -a & \frac{au}{k} & 0 \\ 0 & -u & 0 \\ k & 0 & -u \end{pmatrix}$$

Its characteristic polynomial is

$$f(\gamma) = (\gamma + a)(\gamma + u)^2$$

We find the eigenvector for  $-a$  is  $\begin{pmatrix} u - a \\ 0 \\ k \end{pmatrix}$ , and we find only one eigenvector for the eigenvalue  $-u$ , it is  $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ . This system is defective, we need to find a vector  $\vec{v}$  so that

$$\left( \vec{v} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t \right) e^{-ut}$$

is another solution to the system of equations. To find  $\vec{v}$ , so we solve for

$$(A + uI)\vec{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We find

$$\vec{v} = \begin{pmatrix} \frac{1}{k} \\ \frac{a-u}{au} \\ 0 \end{pmatrix}$$

We now have

$$\begin{aligned} y(t) &= c_1(u-a)e^{-at} + c_3 \frac{1}{k} e^{-ut} \\ v(t) &= c_3 \frac{a-u}{au} e^{-ut} \\ w(t) &= c_1 k e^{-at} + c_2 e^{-ut} + c_3 t e^{-ut} \end{aligned}$$

Using the initial conditions, we find

$$\begin{aligned} c_3 &= v^* \frac{au}{a-u} = y^* \frac{ak}{a-u}, & v(t) &= v^* e^{-ut} \\ c_1(u-a) &= y^* \frac{u}{u-a}, & y(t) &= y^* \frac{ue^{-at} - ae^{-ut}}{u-a} \\ c_2 &= -c_1 k = -v^* \frac{u^2}{(u-a)^2}, & w(t) &= v^* \left( \left( \frac{u}{u-a} \right)^2 (e^{-at} - e^{-ut}) - \frac{au}{u-a} t e^{-ut} \right) \end{aligned}$$

- (b) (10pts) Now, assume that  $a$  is much smaller than  $u$ . Based on the logarithmic plot of the total number of free virus

$$v(t) + w(t) = v^* \left( e^{-ut} + \left( \frac{u}{u-a} \right)^2 (e^{-at} - e^{-ut}) - \frac{au}{u-a} t e^{-ut} \right)$$

how do we estimate the lifetime of infectious virus particles and infected cells, i.e.  $u$  and  $a$ ?

Solution: When  $u \gg a$ ,

$$v(t) + w(t) \simeq v^* \left( \frac{u}{u-a} \right)^2 e^{-at}$$

After an initial shoulder phase,  $v(t) + w(t)$  decays exponentially as  $e^{-at}$ , hence the slope of  $\text{Log}(v(t) + w(t))$  gives us  $a$ , we have then the lifetime of infected cells  $1/a$ .

To find the lifetime of infected cells, we look at the shoulder phase, which is the horizontal distance between the two parallel lines  $\text{Log}(v^* e^{-at})$  and  $\text{Log}(v^* \left( \frac{u}{u-a} \right)^2 e^{-at})$ . The vertical distance is  $\text{Log} \left( \frac{u}{u-a} \right)^2$ . When  $u \gg a$ , we can estimate

$$\text{Log} \left( \frac{u}{u-a} \right)^2 = 2 \text{Log} \left( 1 + \frac{a}{u-a} \right) \simeq 2 \frac{a}{u-a} \simeq 2 \frac{a}{u}$$

Since the slope is  $-a$ , the horizontal distance, length of the shoulder phase is

$$2 \frac{a/u}{a} = \frac{2}{u}$$

So by measure the length of the shoulder phase, we can determine the lifetime of infectious virus particles.