Math153, Fall 2004, Problem Set 1

Due before 4PM on Tuesday October 12th, 2004, at PED office at 1 Brattle Square. No late homework will be accepted. For full credit, please be sure to show and explain your work.

1. Derive from first principles the differential equation associated to the difference equation

$$x_{t+1} = (1+a)x_t.$$

   Show all your work.

2. Analyze the dynamical properties of the non-linear difference equation

$$x_{t+1} = x_t e^{r(1-x_t)}$$

   (a) Decide what the range of x_t should be.

   (b) Decide what the range of r should be in order to preserve the property that x_t increases at low densities and decreases at high densities.

   (c) Find the fixed point of this difference equation.

   (d) Find the range of r so that $p = 1$ is asympotically stable, in the sense that for all x_t close to $p = 1$, x_{t+1} is even closer to p, i.e.

$$|x_{t+1} - p| < |x_t - p|$$

   (e) Let $r = 2.4$, use Mathematic/Matlab/Maple, to plot $y = x$, $y = f(x)$, and $y = f(f(x))$, where

$$f(x) = xe^{r(1-x)}.$$

   Choose the domain of x carefully so that you can see the bifurcation phenonmenon, and label all the fixed/periodic points.

3. Solve the difference equation

$$x_{n+1} = 2x_n + x_{n-1} + 3, \qquad x_0 = -\frac{3}{2} \qquad x_1 = -\frac{1}{2}$$

   The sequence is $-1.5, -0.5, 0.5, 3.5, 10.5, \cdots$

4. Consider the following system of ODE

$$\begin{aligned} \dot{x} &= -x + xy \\ \dot{y} &= y - xy \end{aligned}$$

   where x is the amount of predator, y the amount of prey.

(a) Find all the critical points of this system, and decide the nature of the critical points.

(b) Suppose $x + y = 1$ always. Find ϕ, the average fitness of the entire population, for the frequency depedent selection system

$$\begin{aligned} \dot{x} &= -x + xy - \phi x \\ \dot{y} &= y - xy - \phi y \end{aligned}$$

(c) Using the ϕ obtained above to analyze the behavior of this system.

(d) Suppose intially $x = 1/2$, using the same ϕ, solve for x explicitly.

5. **Protease Inhibitor:** Protease inhibitor of HIV prevent infected cells y from producing infectious virus particles v. Free virus particles, which have been produced before therapy starts, will for a short while continue to infect new cells. The infected cells, however, will produce non-infectious virus particles x. We have the following system of equations to describe the dynamics:

$$\begin{aligned} \dot{y} &= \beta x v - a y \\ \dot{v} &= -u v \\ \dot{w} &= k y - u w \end{aligned}$$

We further assume that x, the amount of uninfected cells, remain constant for the time-scale of investigation at

$$x = x^* = \frac{au}{\beta k}$$

Also assume that

$$y(0) = y^* = \frac{du}{\beta k}, \qquad v(0) = v^* = \frac{d}{\beta}, \qquad w(0) = 0$$

(a) Solve this linear system of equations.

(b) Now, assume that a is much smaller than u. Based on the logarithmic plot of the total number of free virus

$$v(t) + w(t) = v^* \left(e^{-ut} + \left(\frac{u}{u-a} \right)^2 (e^{-at} - e^{-ut}) - \frac{au}{u-a} t e^{-ut} \right)$$

how do we estimate the lifetime of infectious virus particles and infected cells, i.e. u and a?