MATHEMATICS 152, FALL 2004
METHODS OF DISCRETE MATHEMATICS
Outline #10 (Sets and Probability)

Last modified: November 10, 2004

This follows very closely Apostol, Chapter 13, the course pack.

Attachments and Supplements:

- Notes that provide details of some of the examples.
- The "Bayesian Bible," which presents some standard probability problems in unusual contexts.
- The Durango Bill Web pages with bridge and poker probabilities.
- sets.exe and enum.exe, Windows programs available on the course Web site.

1. (This is preliminary material that Apostol takes for granted)
   Given two sets A and B and a universal set S, define union, intersection, complement, and difference. Illustrate your definitions with an example where S consists of the integers 0 through 15, using the program sets.exe, and with an example where S is the set of points in a square region of the blackboard.

2. - Show how to express the intersection and difference of A and B in terms only of union and complement and illustrate the connection using the examples mentioned in the previous item.
   - State Apostol's definition of a Boolean algebra and explain why the closure requirements that he imposes are sufficient to prove closure under difference and intersection. (Apostol, p.471)

3. Define "finitely additive set function," "finitely additive measure," and "probability measure," making it clear what additional requirements are imposed with each new definition.

4. Prove that for a finitely additive set function,

   $$f(A \cup B) = f(A) + f(B - A) = f(A) + f(B) - f(A \cap B)$$

   and illustrate this theorem with a "Venn diagram" where S is the set of points in a square region of the blackboard.
   Prove that for a finitely additive measure f,

   - $f(A \cup B) <= f(A) + f(B)$

- $f(B - A) = f(B) - f(A)$ if $A \subseteq B$.
- $f(A) \leq f(B)$ if $A \subseteq B$.

5. Following the examples in Apostol, section 13.8, explain how to answer the following questions:

   (a) What is the probability of getting exactly two heads when a fair coin for which $P(h) = \frac{1}{2}$ is tossed three times?

   (b) What is the probability of getting a total of either 7 or 11 when tossing two "unloaded" dice?

6. Use induction to prove the following results about probability and counting. Both are so obvious that it takes a minute to realize that they can be proved. The proof is basically the same in both cases.

   - If $A_1 ... A_n$ are mutually exclusive events, the probability that any one of them occurs is the sum of the probabilities for the individual events. (This is equivalent to problem 4 on p. 473 of Apostol.)
   - Consider a set T of n-tuples of the form $x_1, x_2, ..., x_n$. Suppose there are k_1 distinct choices for x_1, k_2 distinct choices for x_2, and so forth. Prove that the number of elements of T is $k_1 k_2 ... k_n$.(Apostol, pp. 482-483)

7. Calculate the probability, when a pair of cards are drawn at random from a single deck of cards, that at least one of them is a spade. Let A be the event that the first card drawn is a spade; let B be the event that the second card drawn(from 51 cards) is a spade. Show that you get the same answer, $\frac{15}{34}$, by each of the following approaches. State in words, and illustrate with a Venn diagram, the reasoning behind each of the four approaches.

   (a) $P(A \cup B) = 1 - P(A' \cap B')$

   (b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

   (c) $P((A \cup B) = P(A) + P(A' \cap B)$

   (d) $P((A \cup B) = P(A \cap B') + P(A' \cap B) + P(A \cap B)$

   In each of the next three items, refer to the four principles of counting as listed in the notes:

   1. Multiply to do sequential counting
   2. Divide to correct systematic overcounting
   3. Divide and conquer
   4. Subtract off special cases

8. In the carnival game "Chuck-A-Luck," you pick a number between 1 and 6. Three fair dice are tossed, and you win if your chosen number appears on one or more dice.
   - Show that your probability of winning is less than 1/2.
   - Determine the probability that your chosen number will appear on 0, 1, 2, or all 3 dice.
   - Show that if you pay 1 dollar to play the game and receive 2 dollars for each occurrence of your chosen number, then the game is fair. (Chuck-A-Luck is discussed in the notes.)
9. Count the number of ways to get each of the following types of 5-card poker hands, using a deck of 52 cards with 4 cards of each of 13 ranks.
   - 4 of a kind(four cards of one rank, the fifth of a different rank)
   - a full house(three cards of one rank, two of another)
   - 3 of a kind(three cards of one rank, two others of different ranks)

   (Poker is discussed in Apostol, section 13.10, in the notes and on the attached Web page)
10. - Count the number of bridge hands with 6 spades, 4 hearts, 2 diamonds, and 1 club.
    - Count the number of bridge hands with 6-4-2-1 suit distribution (6 cards in the longest suit, 4 in the second-longest, 2 in the third-longest)
    - Count the number of bridge hands with 4-4-3-2 or 4-3-3-3 suit distribution, and show that the former has a higher probability by a factor of slightly more than 2.
      (Bridge is discussed in Apostol, section 13.10, in the notes and on the attached Web page.)
11. Define conditional probability and use sets.exe to show examples of how to calculate it. (Apostol, section 13.12)
12. Use the formula for conditional probability to analyze the "bearded man" problem in the notes, the "Paul at Lystra" problem (#1 in the Bayesian Bible) or a similar example of your own invention. It is very useful to arrange the data in a 2-by-2 grid.
13. Use conditional probability to analyze the "math roommate" problem in the notes. (This is very similar to Apostol's Example 2 on p, 488, but two variant versions are also discussed in the notes.)
14. Describe the "Monty Hall problem" and analyze it in terms of conditional probability. (Discussed in the notes and all over the Web. An alternative story line is in #2 of the Bayesian Bible.)

15. Explain how to solve the following problem, which is based on a true story.

    Lisa purchases six Dunkin Munchkins, four plain and two chocolate. She chooses three at random and puts them in a bag for her son Thomas. The other three go into a bag for her daughter Catherine.

    (a) How many ways are there for Lisa to select three of the six Munchkins for Thomas?

    (b) Show that the probability that Thomas's bag has both of the chocolate Munchkins (event A_2) is 0.2.

    (c) Show that the probability that each child has exactly one chocolate Munchkin (event A_1)is 0.6. Explain why $P(A_1) + 2P(A_2) = 1$

    (d) Catherine reaches into her bag and extracts a Munchkin at random. It is a plain one (event B). Show that the conditional probability, given event B, that Thomas has both chocolate Munchkins, precisely one chocolate Munchkin, or no chocolate Munchkins are 0.3, 0.6, and 0.1 respectively.

16. Define independent events (Apostol, section 13.13). Give an example involving dice or using sets.exe of two events that are not independent. Give an example of three events that are independent in pairs but that do not satisfy the criterion for independence of more than two events (Apostol, pp 489-490 and the notes).

17. Explain how to create independent events by means of a "compound experiment," for example a die roll followed by a coin flip, and prove that the criterion for independence is satisfied (Apostol, section 13.15). Use sets.exe to illustrate events that are independent whenever "Random 4 × 4" is used.

18. Define Bernoulli trials, and show that the probability of exactly k successes in n Bernoulli trials is

$$\frac{n!}{k!(n-k)!}p^k q^{n-k}$$

    (Apostol, section 13.17)

19. Define, in terms of 1-to-1 correspondence, what is meant by a countably infinite set. Prove directly from this definition that the collection of all 2-element subsets of the positive integers is countable and that the rational numbers are countable. Illustrate your proof using the Windows program enum.exe from the Web site.

20. Prove that the Cartesian product $P \times P$, where P is the set of positive integers, is countably infinite. The proof is relegated to problem 5 on p.

505 of Apostol, but the hint gives away the answer. There are two equally good approaches to showing that the set of pairs (m, n) is countable:

- Convert (m, n) into the integer $2^m 3^n$, thereby establishing a bijection between the set of pairs (m, n) and a subset of the positive integers. This is the hint for exercise 5 on p. 505 of Apostol.
- Enumerate in succession the pairs for which $m + n$ equals 2,3,4,.... This generalizes the "diagonal" approach used in the preceding topic.

Now prove by induction that the Cartesian product of n countably infinite sets is countable.

21. Prove that a countably infinite union of countably infinite disjoint sets is countable. The proof is relegated to problem 6 on p. 505 of Apostol, but the hint again gives away the answer. The basic idea is that each element of the union is of the form $A_{m,n}$ (the mth member of the nth set) and the set of index pairs is a Cartesian product that was just proved to be countably infinite.

22. Prove that the set of real numbers x satisfying $0 < x < 1$ is uncountable (Apostol, p. 504, example 6).

23. Prove that the collection of all subsets of the positive integers is uncountable. (Apostol, p. 503, example 5)

24. Define probability for a countably infinite sample space. Give an example of a simple probability problem that leads to a countably infinite sample space (for example, rolling a die over and over until a "6" appears), and show that it leads to a convergent infinite series that you know how to sum. (Apostol, section 13.21)