MATHEMATICS 152, FALL 2003
METHODS OF DISCRETE MATHEMATICS
Outline #9 (Group Isomorphisms)

We look at 2×2 matrices over finite fields and discover groups of these matrices that are isomorphic to the symmetry group of the icosahedron and to permutation groups.

No textbook covers this topic. A set of notes is attached.

Material through item 5 is relevant to the quiz on Thursday, November 6

1. Using models of the regular icosahedron and the regular dodecahedron, demonstrate that every rotation permutes the five colors of the edges, and that
   - an element of order 5 corresponds to a permutation like (1 2 3 4 5) that leaves no color fixed.
   - an element of order 2 corresponds to a permutation like (1 2) (3 4) that leaves one color fixed.
   - an element of order 3 corresponds to a permutation like (1 2 3) that leaves two colors fixed.

2. Consider the vector space $\mathbb{F}_4^2$. List the three vectors in each of the five one-dimensional subspaces, and explain how these can be identified with the "committees through the origin" in the medium affine faculty senate. For the sake of concreteness, lay out the senate like this:

education	football	gifts	housing	
Mike$(0, x+1)$	Neil$(1, x+1)$	Owen$(x, x+1)$	Phil$(x+1, x+1)$	diversity
Irma$(0, x)$	Jane$(1, x)$	Kate(x, x)	Lynn$(x+1, x)$	compensation
Emma$(0, 1)$	Fred$(1, 1)$	Greg$(x, 1)$	Herb$(x+1, 1)$	budget
Adam$(0, 0)$	Beth$(1, 0)$	Carl$(x, 0)$	Dave$(x+1, 0)$	athletics

3. Prove that for a 2×2 matrix A , the eigenvalues are determined by the determinant det A and the trace tr A. Then, by considering the four different ways that a 2×2 matrix over $\mathbb{F}_4$ can have determinant 1, count how many matrices in $SL(2, \mathbb{F}_4)$ have trace 0, trace 1, trace x, and trace $x + 1$.

4. By constructing and solving the "characteristic equation" det $(A - \lambda I) = 0$, show that
   - a matrix in $SL(2, \mathbb{F}_4)$ with trace 0 has a single eigenvalue and therefore would have to correspond to an element of A_5 like (1 2) (3 4) with order 2.

- a matrix in $SL(2,\mathbb{F}_4)$ with trace 1 has two eigenvalues and therefore would have to correspond to an element of A_5 like (1 2 3) with order 3.
- a matrix in $SL(2,\mathbb{F}_4)$ with trace x or $x+1$ has no eigenvalues and therefore would have to correspond to an element of A_5 like (1 2 3 4 5) with order 5.

5. With the aid of the program SL2F.exe (on the Web site and also distributed by e-mail), show examples of the action of matrices with order 2, order 3, and order 5 on the one-dimensional subspaces of $\mathbb{F}_4^2$. Interpret each of these as a "collineation" of the medium affine senate that carries instructors into instructors and committees into committees while holding the "origin" Adam fixed.

6. Suppose that G is a group with operation $\times$ and H is a group with operation $*$. Let f be a bijection from G to H that satisfies $f(g \times g') = f(g) * f(g')$ for any pair of elements g and g' in G. The bijection f is called an isomorphism between G and H. Prove the following results.

   - The inverse of f, $f^{-1}: H \longrightarrow G$, is also an isomorphism.
   - f maps the identity of G to the identity of H.
   - f maps inverses into inverses: $f(g^{-1}) = f(g)^{-1}$.
   - If f is an isomorphism from G to H and j is an isomorphism from H to K, then the composition $q = j \circ f$ is an isomorphism from G to K.
   - There is an isomorphism between the rotation group of the icosahedron and A_5, an isomorphism between $SL(2,\mathbb{F}_4)$ and A_5, and therefore an isomorphism between the rotation group of the icosahedron and $SL(2,\mathbb{F}_4)$.

7. Using a diagram of the icosahedron or the dodecahedron (available on a transparency), exhibit explicitly the isomorphism between the rotation group of the icosahedron and $SL(2,\mathbb{F}_4)$. Show how you can predict the effect of performing two successive rotations by multiplying 2×2 matrices.

8. Go back to the coordinatization of the medium affine faculty senate

education	football	gifts	housing	
Mike$(0,x+1)$	Neil$(1,x+1)$	Owen$(x,x+1)$	Phil$(x+1,x+1)$	diversity
Irma$(0,x)$	Jane$(1,x)$	Kate(x,x)	Lynn$(x+1,x)$	compensation
Emma$(0,1)$	Fred$(1,1)$	Greg$(x,1)$	Herb$(x+1,1)$	budget
Adam$(0,0)$	Beth$(1,0)$	Carl$(x,0)$	Dave$(x+1,0)$	athletics

   Demonstrate by example that the bijection that results from interchanging the top two rows and the right two columns (i.e. interchanging x and

$x+1$) is a collineation in the sense that it maps committees into committees. By looking at the five committees of which Adam is a member, show that it corresponds to a permutation like (4 5) which is an element of S_5 but not of A_5.

9. By considering a dodecahedron with six colors used to paint the six pairs of opposite faces (or an icosahedron with six colors or numbers attached to pairs of opposite vertices), show that the rotation group of the dodecahedron (and hence also A_5) is isomorphic to a subgroup of A_6. With the aid of groups.exe, using the tab "Icosahedron 6," give examples of this isomorphism, and indicate what cycle structure corresponds to the elements of order 2, order 3, and order 5 respectively.

10. Enumerate the one-dimensional subspaces of $\mathbb{Z}_5^2$, naming the elements 0, 1, 2, -2, and -1. By considering the five different ways that a 2x2 matrix over $\mathbb{Z}_5$ can have determinant 1, count how many matrices in $SL(2,\mathbb{Z}_5)$ have trace 0, trace 1, trace 2, trace -2, and trace -1. Give a convincing argument that although $SL(2,\mathbb{Z}_5)$ has the same number of elements as S_5, the two groups are not isomorphic. Hint: look at $\left[\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right]$

11. By constructing and solving the "characteristic equation" $\det(A-\lambda I)=0$, show that

    - a matrix in $SL(2,\mathbb{Z}_5)$ with trace 0 has two eigenvalues and therefore would have to correspond to an element of A_6 like (1 2) (3 4) with order 2.
    - a matrix in $SL(2,\mathbb{Z}_5)$ with trace +1 or -1 has no eigenvalues and therefore would have to correspond to an element of A_6 like (1 2 3) (4 5 6) with order 3.
    - a matrix in $SL(2,\mathbb{Z}_5)$ with trace +2 or -2 has one eigenvalue and therefore would have to correspond to an element of A_6 like (1 2 3 4 5) with order 5.

12. Show that the subgroup Z of $SL(2,\mathbb{Z}_5)$ consisting of the identity I and the matrix $-I$ is a normal subgroup. Describe the cosets of Z, show that there are 60 of them, and state a rule for selecting one of the two elements of a coset. Illustrate the action of the quotient group $PSL(2,\mathbb{Z}_5) = SL(2,\mathbb{Z}_5)/Z$ on the one-dimensional subspaces of Z_5^2.

13. The large affine faculty senate can be coordinatized like this:

insurance	junior faculty	football	gifts	housing	
Nancy(-2,2)	Oscar(-1,2)	Kevin(0,2)	Larry(1,2)	Maria(2,2)	compensation
Isaac(-2,1)	James(-1,1)	Frank(0,1)	Gavin(1,1)	Helen(2,1)	budget
Danny(-2,0)	Emily(-1,0)	Alice(0,0)	Betty(1,0)	Chris(2,0)	athletics
Xenia(-2,-1)	Yoric(-1,-1)	Uriah(0,-1)	Viola(1,-1)	Wally(2,-1)	education
Sally(-2,-2)	Terry(-2,-1)	Patty(-2,0)	Quint(1,-2)	Ralph(-2,-2)	diversity

Prove that any matrix in $GL(2,\mathbb{Z}_5)$ (which maps instructors into instructors by acting on their coordinates) also maps committees into committees, but that the subgroup Z that consists of non-zero multiples of the identity maps each committee into itself. (Hint: each committee satisfies a linear equation). Show that the quotient group of $PGL(2, \mathbb{Z}_5) = GL(2, \mathbb{Z}_5)/Z$ has the same number of elements as S_5, that it includes odd as well as even permutations of the six committees to which Alice belongs, and that the cosets containing matrices with determinant +1 and -1 correspond to even permutations while the cosets containing matrices with determinant +2 and -2 correspond to odd permutations.

14. By considering the five different ways that a 2x2 matrix over $\mathbb{Z}_5$ can have determinant 2, count how many matrices in $SL(2, \mathbb{Z}_5)$ have trace 0, trace 1, trace 2, trace -2, and trace -1.

15. By constructing and solving the characteristic equation det $(A - \lambda I) = 0$, show that

    - a matrix in $GL(2, \mathbb{Z}_5)$ with trace 0 and determinant 2 has no eigenvalues and therefore could correspond to an element of A_6 like (1 2) (3 4) (5 6) with order 2.
    - a matrix in $GL(2, \mathbb{Z}_5)$ with trace +1 or -1 also has no eigenvalues and therefore could correspond to an element of A_6 like (1 2 3 4 5 6) with order 6.
    - a matrix in $SL(2, \mathbb{Z}_5)$ with trace +2 or -2 has two eigenvalues and therefore could correspond to an element of A_6 like (1 2 3 4) with order 4.

16. State the Cayley-Hamilton theorem: that any matrix A satisfies its own characteristic equation. From the characteristic equations in the preceding item, show that

    - for determinant 2 and trace 0, A^2 is a multiple of the identity and that A therefore corresponds to a permutation of order 2.
    - for determinant 2 and trace 2, A^4 is a multiple of the identity and that A therefore corresponds to a permutation of order 4.
    - for determinant 2 and trace 1, A^6 is a multiple of the identity and that A therefore corresponds to a permutation of order 6.

    You can use GL2Z5.exe to compute the powers of A in specific cases, but it is more interesting to compute the powers from the characteristic equation.

17. (This is hard) Prove that every collineation of the large affine faculty senate that holds one instructor fixed can be represented by a matrix in $GL(2, \mathbb{Z}_5)$. The proof is an abridgement of the attached proof from Ryan, Euclidean and Non-Euclidean Geometry. What step in the proof fails for the medium affine senate?