MATHEMATICS 152, FALL 2004
METHODS OF DISCRETE MATHEMATICS
Outline #11 (Graphs)

Last modified: December 13, 2004

This is based on Biggs' Chapters 15 and 16 and on excerpts from Grossman and Magnus, Groups and Their Graphs. There are also a couple of pages of notes that were originally prepared for the Summer School course Computer Science S-111.

For many of the proofs requested below, the only thing that is hard is to figure out what needs to be proved! Ask yourself what could go wrong, and devise a proof that rules it out.

The program miles.exe, which will be available in class for demonstrating minimal spanning trees, comes with lots of data files. If you want to run it on your Windows computer, download miles.zip from the course Web site and unzip everything into a single directory. It does lots of other interesting things.

1. Present Biggs' definition of a graph (p. 178). Invent a graph with no more than 5 vertices and no more than 6 edges and represent it by a list of sets, a diagram, and an adjacency list.

   State how the preceding definition might prove inadequate for the following cases and suggest how it could be modified to cover them.

   - "Multigraph:" Vertices represent islands, and edges represent bridges between them. There may be two bridges between the same pair of islands.
   - "Directed graph:"Vertices represent elements of a group, and the edge joining a to b represents the element g such that $ga = b$.
   - "Graph with self loops:" Vertices represent campgrounds in a national park, and edges represent trails that connect them to one another. Then someone builds a "loop nature trail" that starts and ends in the same campground.

2. Define the complete graph K_n (Biggs, p. 179, exercise 3). Draw pictorial representations for K_3 and for K_4 in which edges do not cross. Show that this cannot be done in a plane for K_5, even if the rules are relaxed to allow edges to be represented by curves instead of straight line segments.

3. Define what is meant by isomorphism of graphs (Biggs, section 15.2). Invent an example of two graphs that are isomorphic (even though it may not be apparent from their pictorial representations) and of two graphs that have equal numbers of vertices and edges but that are not isomorphic.

4. Define the degree of a vertex of a graph, and prove that the number of odd vertices is even (Biggs, p. 182).

5. Define the terms "walk," "path," and "cycle" for a graph (Biggs, p. 183 and 184). Define "Hamiltonian cycle" and "Eulerian walk," State and prove a necessary condition for a connected graph to have an Eulerian walk and to have an Eulerian walk which starts and ends at the same vertex (often called an "Eulerian cycle," although it is not necessarily a cycle by Biggs' definition.)

6. (a) Show that for the bridges of Koenigsberg (last page of the attached notes) there is no Eulerian walk. This was the problem that inspired Leonhard Euler, the greatest mathematician of the eighteenth century, to invent the branch of mathematics now known as topology.

   (b) Draw a diagram of the complete graph K_5 (Biggs, p. 179, exercise 3), which has five vertices each with degree 4, and show an example an Eulerian walk.

   (c) Show that the complete graph K_5 has two Hamiltonian cycles that have no edge in common.

7. Define a tree T as a connected graph with no cycles (Biggs, p. 185). Prove the following properties of trees:

   (a) for each pair of vertices x and y, there is a unique path in T joining x and y.

   (b) the graph obtained from T by removing an edge has two components, each a tree.

   (c) $|E| = |V| - 1$.

8. Define "minimum spanning tree" and state and prove the minimum spanning tree property (page 10-18 of the notes). (This is the first part of the proof of Theorem 16.3 in Biggs, but it makes sense to do the proof for an arbitrary set S of vertices.)

9. Describe Prim's algorithm for constructing a minimum spanning tree. Illustrate its action on the graph shown on page 10-19 of the notes, and prove that it is correct. The proof is easy, since the minimum-spanning-tree property has already been proved. (This algorithm, for a non-weighted graph, is described at the start of Biggs, section 16.3, and for a weighted graph it is the "greedy algorithm" that Biggs describes on pp. 200-201.)

10. Describe Kruskal's algorithm for constructing a minimum spanning tree. Illustrate its action on the graph shown on page 10-21 of the notes, and prove that it is correct. The proof is easy, since the minimum-spanning-tree property has already been proved. (This algorithm is not mentioned in Biggs.)

    The next six items are based on the attached excerpts from Grossman and Magnus, Groups and Their Graphs. Most of the rest of this book covers material that is already familiar. If you want to read more, there is one copy in Cabot Library and one in the Math library on the 3rd floor of the Science Center.

11. Draw the graph for a finite cyclic group such as C_4, C_5, or C_6, which has a single generator a. Explain the connection between the group and its graph, with particular attention what features of the graph correspond to the identity I and to the element a^{-1}. Give an example of how different "words", corresponding to different "walks," (the authors use "path" but we will stick to Biggs' terminology) can represent the same group element. (pages 44-47)

12. Draw the graph for the symmetry group of the equilateral triangle, with generators r (a "rotation") and f (a "flip"). Give a nontrivial example of an "empty word" and give examples of cycles in this graph that represent "relations". Explain the convention whereby an edge without an arrow represents a generator of order 2. (pages 48-55)

13. Define the following terms and give an example of each for the cyclic group C_3.

    - "relation" or "generator relation"
    - "defining relation"
    - "equivalent words"
    - "class of equivalent words"

    Show that there are three equivalence classes of words for C_3 (pages 56-63)

14. Show that the defining relations for D_3

    - $r^3 = \mathrm{I}$
    - $f^2 = \mathrm{I}$
    - $rfrf = \mathrm{I}$

    imply that there are precisely six equivalence classes of words. The way to demonstrate this is to prove that an arbitrary word can be rewritten in the form $r^a f^b$ where r is 0, 1, or 2 and b is 0 or 1. Using the graph for D_3 explain the graphical significance of this result. (pages 64-67)

    Note: the final three topics, though important, are somewhat challenging because Grossman and Magnus do not provide the details!

15. List the defining relations for the tetrahedral group A_4. Draw the graph for this group and label each vertex with a member of an equivalence class. Exhibit an isomorphism between the vertex labels and the even permutations of the symbols 1, 2, 3, and 4. (pages 118-119, but the last part is not made explicit).

16. Start with a transparency (provided in class) with the graph for A_5. Show that the permutations $r = (12345)$ and $f = (12)(34)$ satisfy the defining relations for this group. Attach labels to several other vertices of the graph, both as products of generators and as permutations. (pages 167-169)

17. Exhibit a set of defining relations and a graph for the symmetry group of the cube. Here is a description of how to draw the graph shown in the notes.

    (a) Draw a large outer square and a small inner square.

    (b) Join each vertex of the outer square to the nearest vertex of the inner square using two dotted edges, between which is inserted an intermediate square with two of its diagonally-opposite vertices connected to the dotted edges.

    (c) You should now have six squares, all with solid edges. For the outer square, arrows on these edges go counterclockwise. For the inner square and intermediate squares, they go clockwise.

    (d) Join up the remaining vertices of the intermediate squares using four dotted edges. The 12 edges that join different squares (corresponding to the order 2 generator) should all be dotted.

The solid edges correspond to a generator r that satisfies the defining relation $r^4 = 1$. The dotted edges correspond to a generator f that satisfies the defining relation $f^2 = 1$. The third defining relation is $(rf)^3 = 1$. Point out cycles in the graph that correspond to each defining relation.

Using groups.exe, associate a permutation with each vertex of this graph. Choose, for example, $r = (1234)$ and $f = (12)$. Choose a vertex, in the top left corner, say, to correspond to the identity element. Then just keep multiplying by r and f to reach all the other vertices and reconstruct the labeling shown in the notes..