MATHEMATICS 152, FALL 2004
METHODS OF DISCRETE MATHEMATICS
Homework Problems relevant to the third quiz

Last modified: November 10, 2004

Reading

- Notes on Vector Spaces over Finite Fields
- Apostol, Vol. 2, Chapter 13 (the course pack)
- Notes on Probability to accompany Apostol, Vol. 2, Chapter 13

The third quiz will be on Thursday, Dec. 16, not on Dec. 9 as listed in the syllabus.

Required Problems

Problems due Tuesday, Nov. 30. Nothing is due Nov. 23. Don't forget problem 26 from the previous set.

1. With the two-component vectors over $\mathbb{F}_4$ arranged into lines as in the notes, i.e.

   Line 1: multiples of $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.
   Line 2: multiples of $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.
   Line 3: multiples of $\left[\begin{smallmatrix}1\\x\end{smallmatrix}\right]$.
   Line 4: multiples of $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$.
   Line 5: multiples of $\left[\begin{smallmatrix}1\\x+1\end{smallmatrix}\right]$.

   determine the action of each of the following matrices from the group $SL(2, \mathbb{F}_4)$ on the lines, and hence associate a permutation with the matrix. You will learn more if you start by determining the eigenvalues and eigenvectors, but you can also just use a brute-force approach and let the matrix operate on a vector from each of the five lines.

   (a) $\left[\begin{smallmatrix}x & 1\\1 & 0\end{smallmatrix}\right]$
   (b) $\left[\begin{smallmatrix}1 & x\\0 & 1\end{smallmatrix}\right]$
   (c) $\left[\begin{smallmatrix}x+1 & 1\\0 & x\end{smallmatrix}\right]$

2. With the two-component vectors over $\mathbb{F}_4$ arranged into lines as in the preceding problem, construct the matrix of $SL(2, \mathbb{F}_4)$ that represents each of the following permutations:

(a) (13)(25)

(b) (13254)

(c) (132)

Hint: the first and second columns of the matrix in each case are multiples a and b of some vectors on the lines to which lines 1 and 2 respectively are mapped. Use a piece of information about the other lines to set up an equation relating the multiple a for the first column to the multiple b for the second, then choose a and b to make the determinant 1.

3. With the two-component vectors over $\mathbb{Z}_5$ arranged into lines as in the notes, i.e.

   Line 1: multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
   Line 2: multiples of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
   Line 3: multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
   Line 4: multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
   Line 5: multiples of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ if you prefer
   Line 6: multiples of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ if you prefer.

   determine the action of each of the following matrices from the group $PSL(2, \mathbb{Z}_5)$ on the lines, and hence associate a permutation with the matrix. You will learn more if you start by determining the eigenvalues and eigenvectors, but you can also just use a brute-force approach and let the matrix operate on a vector from each of the six lines.

   a. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$
   b. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$
   c. $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$

4. With the two-component vectors over $\mathbb{Z}_5$ arranged into lines as in the preceding problem, let $G = PSL_2(\mathbb{Z}_5)$ be the group acting on $\mathbb{Z}_5^2$ by permuting the lines.

   (a) Write down the four matrices in $SL_2(\mathbb{Z}_5)$ which take Line 2 to Line 6 and Line 1 to Line 4.

   (b) Which two of these matrices take Line 5 to itself? Write down the complete permutation (in S_6) that corresponds to these matrices. To what line do the other two matrices take Line 5?

   (c) Using a generalization of the technique from parts (a) and (b), invent a counting argument to show that $|G| = |A_5|$.

5. A much smaller group than $SL(2, \mathbb{F}_4)$ is $SL(2, \mathbb{Z}_2)$. Write down the six matrices of this group, and show that it is isomorphic to S_3.

6. Show that the permutations
$$a = (15)(24)(36)$$
and
$$b = (14)(26)(35)$$
generate a subgroup of S_6 that is isomorphic to S_3, containing three odd permutations and three even permutations. Hint: think of an equilateral triangle with the vertices numbered 1,2,3 on one side and 4,5,6 on the other.

7. Find a subgroup of 6 matrices in $PSL(2, \mathbb{Z}_5)$ that is isomorphic to S_3. Explicitly identify each matrix in the subgroup with a permutation such as (12) or (123).

8. Find a subgroup of 12 matrices in $SL(2, \mathbb{F}_4)$ that is isomorphic to A_4. Explicitly identify each matrix in the subgroup with a permutation such as (12)(34) or (123). *(Hint: each matrix has the same eigenvector.)*

9. A much smaller group than $SL(2, \mathbb{Z}_5)$ is $SL(2, \mathbb{Z}_3)$. For this group, determine how many elements of order 1, 2, and 3 there are. Show that if you pair up matrices that differ only by an overall sign into cosets, thereby creating $PSL(2, \mathbb{Z}_3)$, you get a group that is isomorphic to the group of symmetries of a regular tetrahedron.

10. Two parts:

    a. Exhibit subgroups of A_5 and of A_6 that are both isomorphic to D_5 and so to one another. (If you use Groups.exe, this will be very easy!)

    b. Exhibit subgroups of $SL(2, \mathbb{F}_4)$ and $PSL(2, \mathbb{Z}_5)$ that are both isomorphic to D_5 and so to one another. *(Hint: the corresponding permutations that you found in part a tell you how each matrix permutes the lines of vectors.)*

Problems due Tuesday, December 7.

11. Apostol, Section 13.4, exercise 1. If no simpler approach occurs to you, try the following:
To show that sets X and Y are disjoint, show that

    - if x is in X it is not in Y
    - and if x is in Y it is not in X.

    To show that sets X and Y are equal, show that

    - if x is in X it is also in Y

- and if x is in Y it is also in X.

12. Apostol, Section 13.4, number 4. To prove the formula by induction,

    - first show that it is true for $n = 2$,

    - then assume that it is true for arbitrary n and show that it is true for $n + 1$.

13. Let the universal set S be the set of all 10000 undergraduates at a university. Here are some subsets.

    - A is the set of 2500 freshmen
    - B is the set of 3000 athletes
    - C is the set of 5000 women
    - D is the set of 200 football players

    There are 1200 female athletes, but none of them play football. Of the freshmen, 1000 are athletes and 1250 are women. 400 are both.

    Express each of the following subsets in terms of A, B, and C, and specify its size.

    (a) The set of female upperclassmen.
    (b) The set of male freshman who are not athletes.
    (c) Translate into the language of sets the statement, "If a freshman plays football, then he is a male athlete."
    (d) An alumnus asks the university to choose a student at random to receive a scholarship. What is the probability that the recipient is either a male freshman athlete or an upperclass woman?

14. Apostol, section 13.9, exercise 10.

15. Apostol, Section 13.11, #7 and #8. In part c, you need to know that ace, 2, 3, 4, 5 in a suit counts as a straight flush.

16. A chance device used by the Lottery Commission can generate any number between 2 and 30 for the "daily numbers game", with the probability of any individual number determined by a secret formula.
    Event A is "the number is prime," and P(A)= 0.4.
    Event B is "the number is less than 15," and P(B) = 0.5.
    Event C is "the number is a prime less than 15," and P(C) = 0.3.

    (a) Are events A and B independent? Explain.

        For each of the following events, specify the event in terms of A and B, and calculate its probability.

(b) Event D: "the number is a prime greater than or equal to 15."
(c) Event E: "the number is either prime or less than 15, or both."
Suppose that today's number has been generated.
(d) If it is known to be prime, what is the probability that it is also less than 15?
(e) If it is known to be greater than or equal to 15, what is the probability that it is not prime?

17. (This problem is loosely based on some work with a recently-discovered 1774 census of Rhode Island)
A genealogist, on analyzing names in 18th-century Rhode Island, has ascertained the following:

- The probability that a male child was a slave was 0.3.
- Male children who were slaves were given classical names like "Caesar" and "Aesop" with probability 0.6.
- Male children who were free were given classical names like "Caesar" and "Aesop" with probability 0.2.

Event A is "the child was a slave" and event B was "the child had a classical name."

(a) Using set-theoretic notation, express the event "the child was either free or had a classical name" in terms of events A and B and calculate its probability.
(b) Are events A and B independent? Justify your answer.
(c) The genealogist encounters the classical name "Cicero Greene". What is the conditional probability, given his name, that Cicero was a slave?
(d) The genealogist learns that Cicero had one sibling, his brother Roger. Either both were slaves, or both were free. What is the conditional probability that both were slaves?

18. In an eight-team football league there are four referees. At the start of the season coins are distributed to them at random. Three of the coins are fair ones, but the fourth has two heads. Event A is that referee R receives the two-headed coin. An astute coach notices that the first three coin flips of referee R have all come up heads. This is event B. The coach proposes to say to the referee, "Are you using a two-headed coin?" but he wants to know the probability of Event A in order to be sure that he has a good chance of being correct.

Calculate the conditional probability, given event B, that referee R is using the two-headed coin.

19. (a) Determine how many ways there are to select a subset of 5 of the 13 spades in a deck of cards.You may leave your answer in terms of products of integers, but expand any factorials or binomial coefficients.

    (b) Determine how many distinct bridge hands contain 5 spades, 3 hearts, 3 diamonds, and 2 clubs. As above, you may leave your answer in terms of products of integers.

    (c) Determine how many distinct bridge hands have 5 cards in the longest suit, 2 cards in the shortest suit, and 3 cards in the other two suits.

    Problems due Tuesday, December 14.

20. The Queen of Sheba has come to Jerusalem to find a prophet. There are two sorts of prophets: true prophets, who speak the truth nine times out of ten, and false prophets, who speak the truth half the time. Prophet agents are forbidden to reveal explicitly which of their prophets are true ones.

    The queen wants to be more than 90% certain that the prophet she selects is a true one. She hires a prophet agent who brings out three prophets: two true ones and a false one. "2 out of 3 – that's not good enough, is it?" the agent asks. "Sure it is," says the queen, "as long as I can ask one yes-no question." "Ask away," says the agent.

    The queen asks prophet 2, "Is prophet 3 a true prophet?" On hearing the answer she makes her selection and heads home. How did she do it?

    Hint: Event B is "answer is yes", B' is "answer is no." Events A1, A2, A3 are respectively "prophet 1 is a true prophet," etc.

    The queen must be sure that for either answer, her selection (made after hearing the answer) satisfies $P(A|B)$ (or $P(A|B')$) $> .9$.

    For a more fanciful version of this problem, see #3 in the Bayesian Bible.

21. In the admissions office at Monty Hall University there are four interviewers. Three of them, Fl, F2, and F3, are friendly, while the fourth, U, is unfriendly. Every morning the Dean of Admissions assigns them randomly to offices 1, 2, 3, and 4, with an equal probability for each possible assignment. A student arrives for an interview and is asked to select which office he wants to be interviewed in. He chooses office 1 and learns that the interviewer in there is busy for the next half hour. "While you are waiting," says the Dean to the student, "I would like you to meet one of our friendly interviewers. From offices 2, 3, and 4, I will choose the lowest-numbered friendly interviewer." He opens the door of office 2 and introduces the student to an interviewer.
    This is Event B – the lowest-numbered available friendly interviewer was

in office 2.
Event A is that office 1 contains the unfriendly interviewer.

(a) Enumerate all the ways of assigning interviewers to offices that lead to Event B. Assign a probability to each, and show that the sum of these probabilities equals the probability of Event B.
(b) Given that Event B has occurred, determine the conditional probability of event A.
(c) What is the probability that the friendly interviewer to whom the student was introduced by the Dean was F1?

22. Apostol, section 13.14, exercise 14.

23. At Awesome State University, grade inflation has become so extreme that the NCAA requires athletes to get A's in at least half their courses to remain eligible. A star football player can choose to take four courses ($p = 1/2$ for an A) or just three courses ($p = 2/3$ for an A). Which choice gives the higher probability of remaining eligible?

24. You have to deliver crucial supplies using airplanes with very unreliable engines. Each engine has a probability p of lasting for the entire flight, and engine failures are independent events. If half or more of the engines fail, the plane crashes. Your choice is between using two-engine planes, which crash if either engine fails, or 4-engine planes, which crash if two or more engines fail.

    (a) What is the probability that exactly three engines on a four-engine plane will survive?
    (b) Determine for what value of p the probability that a plane will not crash is the same for 2-engine and 4-engine planes.
    (c) For this value of p, would 6-engine planes be a better choice?

25. Boxcar Bob owns three dice. Two of them are unloaded, but the third has $p = 1/2$ for a 6, and $p = 1/10$ for the numbers 1 through 5. Event B is that he rolls a randomly chosen die three times, and a 6 appears precisely once. Event A is that Bob is using the loaded die.

    (a) Calculate $P(A \cap B)$, $P(A' \cap B)$, and $P(B)$.
    (b) Given event B, what is the conditional probability that Bob is using the loaded die?

26. Apostol, section 13.18, exercise 14.

27. Apostol, page 508, section 13.23, exercise 8.

28. The Afghan warlord General Dostum has captured a female storyteller named Bernoulli (like most Afghans, she uses only one name). She offers to tell him a story every day, on condition that he will not turn her over to the CIA that day if her story makes him laugh. Since Dostum has a fine sense of humor, the probability that a story will NOT make him laugh is $p = 1/5$.

    (a) What is the probability that she is turned over to the CIA after telling precisely 3 stories?

    (b) What is the probability that she is turned over to the CIA on or before the third day?

    (c) Donald Rumsfeld is asked by a reporter, on the day of her capture, "What day is the CIA most likely to gain custody of Bernoulli?" What is his correct answer? Explain.

29. Ill-tempered pooch Fido greets the postal delivery person with a growl every day, but he only bites him or her with probability p. In Fido's town, the legal maxim "every dog gets one bite" is honored, but the second time that Fido bites, it is off to the pound with him.

    Find a formula for the probability P_n that Fido delivers his second and final bite on day n. (Hint: the first bite could have been delivered on any of the $n-1$ previous days).

30. An ambitious preschool, whose aim is to prepare toddlers for Harvard, has 8 students, 4 boys and 4 girls. Following the example of Harvard, it divides its students randomly into two groups of 4, which it names "Lowell House" and "Eliot House."

    (a) What is the probability P_4 that all four boys end up in Eliot House?

    (b) What is the probability P_3 that three boys and one girl end up in Eliot House, with the other boy and three girls in Lowell House?

    (c) What is the probability P_2 that optimal diversity is achieved, with two boys in Eliot House and two in Lowell House?

    (d) Verify that the sum of probabilities for all the ways of dividing the class between the houses is correct.

    (e) A newly hired teacher meets one student, chosen at random, from Eliot House. The student is a boy. Calculate the conditional probabilities, given this event, that there are 4, 3, 2, 1, or 0 boys in Eliot House. (These conditional probabilities should also sum to 1.)

    (f) Next the teacher meets a randomly chosen student from Lowell House, who turns out to be a girl. Calculate the conditional probabilities, given both this event and the previous one, that there are 4, 3, 2, 1, or 0 boys in Eliot House.

31. Apostol, section 13.20, exercises 9a, 9c, and 10b. For the second of these, use the facts that every interval of positive length contains at least one rational number and that the rational numbers are countable.

32. The U.S. Supreme Court, in a unanimous decision, has decreed that all future high-school graduates will be given serial numbers and that every university must immediately submit an acceptance list of the serial numbers of all the students they will ever admit. The Harvard Admissions Committee convenes for one last time to agree on its acceptance list. An amateur programmer on the committee demonstrates a program P he has written, which produces output that starts like this:

    Possible acceptance lists for Harvard College – choose one
    1: 2, 3, 5, 8, 13, 21, 34, ...
    2: all even-numbered students
    3: all students with prime serial numbers
    4: ....

    "All we have to do is select one of the lists that my program can produce," he announces, "and our job is done forever. My program won't waste your time, since every possible list will be produced once and only once."

    "Wait a minute," says a mathematician on the committee. "I have another idea for an acceptance list – call it list A. Since your list 1 rejects student 1, my list A will admit him. Since your list 2 admits student 2, list A will reject him. In general, for student k, we will run your program and see if your list k has student k on it. If not, my list A will accept him."

    "I like the idea of your list A," says a sociologist, "since it counteracts any discrimination that may unwittingly be built into the computer program. But if we just run the program for a while, it should print out list A, say as list number n. Then we just tell the Dean that we've chosen list number n and we're done. I'm curious to know whether student number n will be accepted or not."

    (a) On the basis of what you know about countability and one-to-one correspondence, explain whether it is possible for program P to output all possible acceptance lists.

    (b) Prove your answer to part a by giving a careful (paradoxical) answer to the sociologist's question about whether student number n is accepted according to list number n.

Exploratory Problems

1. Find a 10-element subgroup of $SL(2, \mathbb{F}_4)$ that includes $\left[\begin{smallmatrix} 0 & 1 \\ 1 & x \end{smallmatrix}\right]$ and is isomorphic to D_5, the group of symmetries of the regular pentagon. *(Hint: Consider the* 180° *rotation about an axis in the equatorial plane that carries* $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ *into itself. You know what this does to* $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$*, so you can write down the matrix. That matrix, along with* $\left[\begin{smallmatrix} 0 & 1 \\ 1 & x \end{smallmatrix}\right]$ *generates the group.)*

2. Find a 12-element subgroup of $SL(2, \mathbb{F}_4)$ that includes $\left[\begin{smallmatrix} x & x+1 \\ x+1 & x \end{smallmatrix}\right]$. *(Hint: This matrix has one eigenvector, and there are 11 others with the same eigenvector: the identity, two more of order 2, and eight of order 3.)*

3. Using the same approach as in the notes, determine how many elements are in the group $SL(2, \mathbb{Z}_7)$, then show that the group $PSL(2, \mathbb{Z}_7)$ has 168 elements. Determine how many of these elements have order 7.

4. Conway's atlas of groups claims that $PSL(2, \mathbb{F}_9)$ has 360 elements and is isomorphic to A_6. Show that each of these groups has 360 elements, and show that each group has the same number of elements of each order.

5. Find a 6-element subgroup of $SL(2, \mathbb{F}_4)$ that includes $\left[\begin{smallmatrix} x & 1 \\ 0 & x+1 \end{smallmatrix}\right]$ and is isomorphic to S_3, the group of symmetries of the equilateral triangle. *(Hint: One way to approach this is to use the diagram attached to the notes. The given matrix is a rotation about one of the vertices of the top pentagon, involving the three vectors* $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 \\ x+1 \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right]$. You need the three elements of order 2 to complete the subgroup.)

6. Find a matrix B with the property that for any matrix A in $SL(2, \mathbb{Z}_2)$ (problem 5), the matrix BAB^{-1} is the corresponding element of the subgroup of $SL(2, \mathbb{F}_4)$ that you found in the preceding problem. *Hint: matrix B carries one axis of rotation into the other.*

7. With the preceding problem for inspiration, find a subgroup of $PGL(2, \mathbb{Z}_5)$ that is isomorphic to S_3. It is sufficient to specify one of the four matrices in each coset, but remember that when you "test for equality," being equal up to an overall multiple is good enough. Use the same numbering of lines as in the notes, i.e.

   Line 1: multiples of $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$

   Line 2: multiples of $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$

   Line 3: multiples of $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$

   Line 4: multiples of $\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$

   Line 5: multiples of $\left[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]$ if you prefer

   Line 6: multiples of $\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}\right]$ if you prefer.

A permutation specifies the image of line 1 and of line 2, so up to a multiple you know each column of the desired matrix. For the odd permutations you want a trace of zero and a determinant of +2 or -2. For the the even permutations you should get a determinant of +1 or -1 and a trace of +1 or -1. You can save time by letting GL2Z5.exe calculate the effect of the matrices on the lines.

8. For a matrix in $SL(2, \mathbb{Z}_7)$, find a value of the trace that will lead to an element of order 7. It's not sufficient just to show that there are no eigenvalues – use the Cayley-Hamilton theorem to show that A^7 is a multiple of the identity (slightly tedious calculation).

9. Apostol, section 13.4, exercise 10. In part d, the book is wrong. The symmetric difference of A and B is not disjoint from A or from B. However, it is disjoint from $A \cap B$, which is what you should prove instead for that part of the problem.

10. Apostol, section 13.7, exercise 16.

11. Poker novice Jane picks up her five cards and asks "What did you say the probability was for event A (no two cards of the same rank)?" Veteran Betty tells her. Jane then says "Well, event B (all four suits represented in the hand) has just occurred for me. Is that worth anything?" Betty says, "No, but given that, do you want to know the conditional probability for A, which I'm planning to use when I bet against you?"
Calculate $P(A)$, $P(B)$, $P(A \cap B)$, and $P(A|B)$. You'll want a calculator.

12. The last problem from the "Bayesian Bible" (attached to the probability outline, and on the Web site).

13. (At Marcia Weis's invitation, I once presented the solution to this problem at a bridge lecture on board the Grand Princess.)
The year is 2202, and Earth can support its large population only by having billions of people on cruise ships. Marcia Weis, the bridge director for Megaprincess Lines, is preparing a duplicate bridge tournament for about 10 million players. In the opening deal, she arranges for the North-South pairs all to have the same hands, including 9 spades – all but the queen, 4, 3, and 2. She then distributes the remaining 26 cards (4 spades and 22 others) in every possible way between East and West. Thus every declarer, in trying to guess how the missing cards are divided between East and West, will know that each possible outcome will occur precisely once on a cruise somewhere.

    In what follows, let $N_22 = \frac{22!}{(13!11!)} = 4522$ and express all answers as a multiple of N_22.

    (a) How many different East hands of 13 cards can Marcia prepare?

(b) In how many of these hands does East have all four of the missing spades?

(c) In how many of these hands does East have all of the missing spades except the queen?

(d) In how many of these hands does East have precisely three of the missing spades?

(e) In how many of these hands does East have precisely two of the missing spades? (As a check, add together this number, twice the answer to part b, and twice the answer to part d. This should agree with the answer to part a).

(f) Since Princess passengers have been taught the maxim 'Eight ever, nine never" at the mandatory lifeboat drill, they will all play this deal the same way, by leading the ace and king of spades and hoping that East or West will be forced to play the queen. This approach will succeed if East and West each has two spades or if either East or West has only the queen while the other has the remaining three spades. In how many of the hands will the approach succeed? What is the probability of success?

(g) Suppose that you are North, you lead the ace of spades, and both East and West play their lowest spade, which is not the queen. What is the conditional probability that East and West now each has one of the missing spades? What is the conditional probability that East has both of the missing spades?

14. Apostol, Section 13.23, page 509, exercise 12.

15. Apostol, Section 13.20, exercises 7 and 8.

16. Legendary baseball manager Bill Veeck once sent a midget to the plate in hopes that he could draw a walk. The midget had instructions to take pitches until he walked or struck out, and of course because of his small strike zone, the probability p that the pitcher would throw a ball was fairly high. Calculate, as a function of p, the probability that the midget will walk: that he will receive 4 balls before 3 strikes. Although there is no closed-form solution to this problem, you can get the answer as a sum of three terms in two different ways, and both should lead to the same polynomial in p.

    (a) Add together the probability for the midget to walk on 4, 5, or 6 pitches, taking 0, 1, or 2 strikes before the final ball.

    (b) Assume that the pitcher throws 6 pitches, at which point the midget has struck out if he has not walked. Add together the probability for 4, 5, or 6 of them to be balls.

You may want to evaluate the answer for a few values of p, but you are not required to. If you do, consider using Mathematica.