MATHEMATICS 152, FALL 2004
METHODS OF DISCRETE MATHEMATICS
Homework Problems relevant to the first quiz

Last modified: September 12, 2004

Reading

- Biggs, Section 27.3 (Symmetry)
- Biggs, Sections 5.2-5.4, 10.6, 12.5, 12.6, and 21.1 (Permutations)
- Biggs, Chapter 20. (Groups)
- Biggs, Sections 13.1–13.3 (Modular Arithmetic)

Required Problems

1. Determine the number of symmetries of a regular n-gon. Specify how many are rotations about an axis perpendicular to the plane containing the n-gon and how many are reflections (or you can think of them as rotations) about an axis in the plane of the figure. Treat the identity as a separate special case.

2. Build physical models of the five regular polyhedra, known as the Platonic solids. (Bring these to class on Thursday, September 23, and just include a note on your homework stating that you have done them)

3. Make a table listing the symmetries for each of the five regular polyhedra (tetrahedron, cube, octahedron, dodecahedron, icosahedron), and classify them by type. *(This is just a summary of results from the first class.)*

4. If you number the vertices of a tetrahedron as 1, 2, 3, 4, then any symmetry of the tetrahedron is a permutation of the set $\{1,2,3,4\}$

   (a) List the permutations that correspond to rotations through angles of 120 or 240 degrees.

   (b) List the permutations that correspond to rotations through an angle of 180 degrees.

   (c) Show that any these permutations can be written as the product of two transpositions.

5. Here is a permutation of the set {1,2,3,4,5,6,7,8,9}, expressed as a function.

x	1	2	3	4	5	6	7	8	9
$a(x)$	4	6	3	8	7	2	9	1	5

Express a in cycle notation.

6. Two permutations of the set $S = \{1,2,3,4,5,6\}$ are

$$a = (135)(26)$$
$$b = (1246)$$

(a) Use the "pure cycle" approach, as described in the notes, to express ba and ab in cycle notation.

(b) Make tables showing the effect of a, b, ba, and ab as functions from the set S to itself.

7. Let $\sigma = (1234)$ and $\tau = (123)(45)$ be two permutations of the set of 5 elements. The inverse of a permutation "undoes" the effect of that permutation, so that, for example, $\sigma^{-1}\sigma$ is the identity. Compute σ^{-1}, τ^{-1}, $\sigma\tau$, and $\tau\sigma$, and write each of σ and τ as a product of transpositions. (A transposition is a permutation that just interchanges two elements, like (12).)

8. Let $\sigma = (1357)(246)$ be a permutation of the set of seven elements.

(a) Find σ^2, σ^3, and σ^{-1}.

(b) What is the smallest positive integer n such that σ^n is the identity? (This n is defined to be the *order* of the element σ.)

9. In the group S_6, how many cycles are of the form (12)(34)(56)? In the group S_7, how many cycles are of the form (12)(34)(56)(7)?

10. Make a table of compositions/products for the elements of the symmetry group of the square. *(Hint: Part of this exercise is to determine a good notation for the symmetries.)*

11. Show that if G is a group and $x, y \in G$, then $(xy)^{-1} = y^{-1}x^{-1}$.

12. One labeling of the regular hexagon is the figure below, where any symmetry operation on the hexagon permutes the sets of elements that share a number. For example, the three edges numbered 4 may be interchanged with the three edges numbered 5.

   (a) Write the five permutations from S_5 that represent non-trivial rotations about an axis perpendicular to the plane of the hexagon.
   (b) Write the three permutations from S_5 that represent reflections about axes between two opposite vertices (or, if you prefer, 180° rotations about theses axes).
   (c) Express the last three non-trivial symmetries of the regular hexagon as permutations from S_5, and describe what each does geometrically.

13. Consider the following six functions:

$$\begin{aligned} f_1(x) &= x & f_2(x) &= 1-x \\ f_3(x) &= \tfrac{1}{x} & f_4(x) &= \tfrac{1}{1-x} \\ f_5(x) &= \tfrac{x}{x-1} & f_6(x) &= \tfrac{x-1}{x} \end{aligned}$$

Now think of these as a group under the operation composition. For example,

$$\begin{aligned} (f_2 \circ f_3)(x) &= f_2(f_3(x)) \\ &= f_2(\tfrac{1}{x}) \\ &= 1 - \tfrac{1}{x} \\ &= \tfrac{x-1}{x} \\ &= f_6(x). \end{aligned}$$

Make a table of compositions for this group.

14. Given a set G with a binary operation (denoted by adjacency of elements) that is closed under the operation and that satisfies the three following axioms:

   G1'. Given any $x, y, z \in G$, $(xy)z = x(yz)$.

   G2'. Given any $a, b \in G$, there is a unique element $x \in G$ such that $xa = b$.

   G3'. Given any $a, b \in G$, there is a unique element $y \in G$ such that $ay = b$.

   Show that G is a group under this operation.

15. In $\mathbb{Z}_{13}$

   (a) what is the additive inverse of [5]?

   (b) for what m and n does $13m + 5n = 1$?

   (c) what is the multiplicative inverse of [5]?

   (d) what is the square root of [-1]?

16. In $\mathbb{Z}_{12}$

   (a) which elements have no multiplicative inverse?

   (b) what is the multiplicative inverse of each of the remaining elements?

   (c) Prove that the invertible elements form a group, and write out the multiplication table for this group.

17. Write out the group multiplication table for the invertible elements of $\mathbb{Z}_{18}$ under multiplication.

18. In class or in the preceding problem, we have discussed the following seven groups, all of which have six elements:

   - the symmetry group of the equilateral triangle, D_3
   - the permutation group on a set of three symbols, S_3
   - the group of additive congruences modulo 6, $(\mathbb{Z}_6, \oplus)$
   - the multiplicative group of non-zero congruences modulo 7, $(\mathbb{Z}_7^\times, \otimes)$
   - the six rotations of a regular hexagon about an axis perpendicular to the hexagon, C_6
   - the six functions from problem 13 above under composition, G_13.
   - the invertible elements of $\mathbb{Z}_{18}$ under multiplication, G_18.

There are fundamentally only two different groups here, one of them abelian and cyclic (C_6) and one not (S_3). For each case, make a table in which the columns are the "groups" as labeled above and the rows are the elements. The elements in each row should be "the same" in the sense that if the element in row i times the element in row j equals the element in row k for one column, this property holds for all columns in the table. This matching-up of group elements is known as an "isomorphism," and this concept will become important later in the course.

19. Show that [3] is a generator for the multiplicative group $\mathbb{Z}_7^\times$, and find any other generators. (For the definition of generator, see Biggs, Section 20.6.)

20. Find the order of every element of $\mathbb{Z}_{13}^\times$. In particular, show that $\mathbb{Z}_{13}^\times$ is cyclic, and find all of its generators.

21. Find subgroups of the symmetry group of the icosahedron isomorphic to D_3, D_5, and A_4, and for each, find two permutations that generate the subgroup. (You can check your answer using Groups.exe, but find different subgroups than the ones given by clicking the "Show ..." buttons in the program.)

22. Find all subgroups of D_3 and D_4.

23. Let $G = D_4$, where one rotation by $\frac{\pi}{2}$ is denoted a and one reflection is denoted b, so that we may write $G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. Let $H_1 = \{e, b\}$ be the subgroup consisting of the identity and one reflection, and let $H_2 = \{e, a^2\}$ be the subgroup consisting of the identity and the rotation by π. Find all left-cosets and all right-cosets of H_1 and H_2.

24. Show how we may consider S_3 to be a subgroup of S_4, and find all of its left-cosets. Give a geometric interpretation of this in terms of the cube and the equilateral triangle.

25. Consider the symmetry groups of the tetrahedron, the cube, and the icosahedron. For each case, make a table listing all orders of subgroups that occur. For a subgroup of that order, show in your table the number of elements in a left-coset and the number of left-cosets. Remember that a subgroup of G could have order 1 or could be G itself.

26. Let $G = A_5$, and consider the subgroup $H_1 = \langle(123)\rangle$ (the subgroup generated by (123).

    (a) Find the subgroup of G conjugate to H_1 under the element $g = (14)(25)$.

    (b) Find the subgroup of G conjugate to H_1 under the element $g = (12)(45)$.

(c) Is H_1 self-conjugate? Explain.

Exploratory Problems

1. Find the coordinates in $\mathbb{R}^3$ for the vertices of the cube, octahedron, and tetrahedron in reasonable orientations.

2. Discuss the distinction between the symmetry groups of planar objects and those of solid objects. In particular, why are reflections considered valid symmetries of planar objects, but not of solid objects? (What *is* a reflection of a solid object?) How would the symmetry groups be different if we did/did not allow reflections?

3. Describe the symmetry group of the circle.

4. Given a set S with a binary operation (denoted by adjacency of elements) that is both closed and associative, suppose we have the following axiom in place of our usual identity axiom:

   S3.' Given $x \in S$, there is an element $e \in S$ (possibly depending on x) such that $xe = ex = x$.

   For example, S could be the set of 2×2 matrices of the form $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$with $ab = 0$. If x is the matrix $\left[\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right]$, then e is the matrix $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

   There is no inverse axiom, since we are not necessarily considering a group.

   Decide whether or not the usual identity axiom holds. Either prove that it does, or invent a counterexample.

5. Biggs, Exercises 20.4, number 3. "Show that M is a group" means "show that all the group axioms are satisfied."

6. Consider the set $\mathbb{Q}^\times$ of non-zero rational numbers, and define the binary operation * as follows:

$$x * y = \begin{cases} xy & , \text{ if } x, y > 0 \\ xy & , \text{ if } xy < 0 \\ 2xy & , \text{ if } x, y < 0 \end{cases}$$

   That is, if x and y are both positive, then $x * y$ is the usual product, and if one of x and y is positive, then $x * y$ is also the usual product, but if both x and y are negative, then $x * y$ is twice the usual product.

   Verify the four group axioms for $(\mathbb{Q}^\times, *)$.

7. Let G be a finite group (that is, $|G| < \infty$), and let $g \in G$. Prove that the order of g is finite. (This is a pretty obvious result, but you must make sure that your proof relies on nothing but the group axioms.)

8. Find all generators of the cyclic group C_n. *(Hint: Try the case when n is prime first.)*

9. You can associate the vertices of a square with the four vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$v_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$v_3 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$
$$v_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Now the matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, a clockwise rotation that (in the alibi approach) carries v_1 into v_2 and v_2 into v_3, represents the permutation (1234). Find matrices that represent each of the following permutations:

   (a) B represents (13)(24)
   (b) C represents (12)(34)
   (c) D represents (13)
   (d) E represents (1432)

Calculate the products DC and CD and show that the result is consistent with the effect of multiplying the permutations that these matrices represent.

10. In the same manner as the preceding problem, invent three vectors that represent the three vertices of an equilateral triangle. Write down six matrices that correspond to the six elements of S_3, and give two examples of how multiplication of matrices is consistent with multiplication of permutations. The first of these should involve the product of two reflections, which correspond to odd permutations like (12). The second should involve the product of a rotation and a reflection.

11. (a) Does [15] have a multiplicative inverse in $\mathbb{Z}_{36}$? Explain.
   (b) Use Euclid's gcd algorithm to find an element $[x]$ in $\mathbb{Z}_{36}$ for which $[15][x] = [3]$.
   (c) Find an element $[y]$ in $\mathbb{Z}_{36}$ for which $[15][y] = [27]$.