MATHEMATICS 102, FALL 2002
METHODS OF DISCRETE MATHEMATICS
Notes on Probability to accompany Apostol, Vol. 2, Chapter 13

1 Combinatorial analysis

The Cartesian product $A \times B$ of sets A and B is the set of ordered pairs a, b where $a \in A, b \in B$. If A has n_a elements and B has n_b elements, $A \times B$ has $n_a n_b$ elements.

Example: a die roll (1 ...6) followed by a coin toss(H or T): there are 6×2 elements: 1H, 2H,...6H, 1T,...6T.

- Principle 1: Multiply for sequential counting. If we are forming a set of n-tuples where there are k_1 choices for the first element x_1, k_2 for x_2, and so on, the number of n-tuples $x_1 x_2 ... x_n$ in the set is $k_1 k_2 ... k_n$
- Principle 2: Divide to correct systematic overcounting. If M different n-tuples all correspond to the same element in our sample space, count the n-tuples and then divide by the number of times M that each was overcounted.
- Principle 3: Divide and Conquer. To count the number of elements in a union of disjoint subsets, count each subset and sum the results.
- Principle 4: Subtract off special cases. To count the number of elements in a difference $A - B$ where $B \subset A$, count each set and take the difference.

Classic example of principles 1 and 2: counting the number of k-element subsets of an n-element set.

First count the k-tuples whose elements are all different:

$$n(n-1)(n-2)...(n-k+1)$$

. Then divide by $k!$ to correct for overcounting.

2 The carnival game called "Chuck-A-Luck"

You pick a number from 1 to 6. Three fair dice are rolled. If any comes up with your number, you win. What is the probability of winning?

Use principle 4, where A is the set S of all possible rolls and B is the set of winning rolls, so $P(B) = 1 - P(B')$ where B' is the set of losing rolls.

The number of losing rolls is 5 x 5 x 5 = 125.
The total number of rolls is 6 x 6 x 6 = 216
So there are 216-125 = 91 winning rolls, and your chance of winning is 91/216.

Sanity check:

- 1 roll with 3 of your number
- 15 rolls with 2 of your number (3 places for the other number x 5 choices for it)
- 75 rolls with 1 of your number (3 places for it, and 5 x 5 pairs of other numbers for the other two dice)
  $3 \times 1 + 15 \times 2 + 75 \times 1 = 108$, which is half of 216.

3 Examples from poker

How many distinct 5-card poker hands can be dealt from a 52-card deck?
Answer: Select 5 cards sequentially: the number of ways to do this is $52 \times 51 \times 50 \times 49 \times 48$
But this generates each distinct hand $5! = 120$ times.
So the number of distinct hands is $\frac{52!}{47!5!}$

Some more examples from poker where multiplication leads to the answer:

- How many distinct ways are there to get 4 of a kind?
  13 choices for the rank of the 4, 48 choices for the other card.
- A full house (3 of one rank, 2 of another)
  13 choices for the first rank (with 3 cards), 4 for the suit that is missing.
  12 choices for the second rank (with 2 cards), 6 for the pair of suits that have the second rank.
- Three of a kind (3 of one rank, the other two do not match)
  13 choices for the first rank (with 3 cards), 4 for the suit that is missing.
  48 choices for the fourth card
  44 choices for the fifth card (because 2 ranks are now excluded)

4 Examples from bridge

- How many ways are there to select 6 cards from the 13 spades?

$$\frac{13!}{7!\,6!}$$

- How many distinct hands have 6 spades, 4 hearts, 2 diamonds, 1 club?
  Apply the same analysis to each suit in turn:

$$\frac{13!}{7!\,6!}\frac{13!}{9!\,4!}\frac{13!}{11!\,2!}\frac{13!}{12!\,1!}$$

- How many distinct hands have a 6-4-2-1 distribution?
  Multiply the preceding number by the number of ways to choose the 6-card suit, the 4-card suit, etc, which is 24.

- How many distinct hands have a 4-4-3-2 distribution?
  Start with
  $$\frac{13!}{9!\,4!}\frac{13!}{9!\,4!}\frac{13!}{10!\,3!}\frac{13!}{11!\,2!}$$
  Now there are 4 ways to select the 2-card suit followed by 3 ways to select the 3-card suit, so multiply by 12.
  (Alternative: multiply by 24 as above, but divide by 2 to correct for overcounting)

  How many distinct hands have a 4-3-3-3 distribution?
  $$\frac{13!}{9!\,4!}\frac{13!}{10!\,3!}\frac{13!}{10!\,3!}\frac{13!}{10!\,3!}$$
  Now there are 4 ways to select the 3-card suit, so multiply by 4.
  (Alternative: multiply by 24 as above, but divide by 6 to correct for overcounting)

5 Conditional probability

Suppose that A and B are events (subsets) of a sample space S. If we know that event B has occurred, what is then the probability that A has also occurred?

Example: I roll two dice and note that the sum is six. What is the probability that

- at least one 2 is showing?
  Answer: there are 5 pairs in set B: (1,5), (2,4), (3,3), (4,2), and (5,1). Two of them belong to set A so the probability is $\frac{2}{5}$

- at least one 3 is showing?
  Answer: there are 5 pairs in set B: (1,5), (2,4), (3,3), (4,2), and (5,1). Only one of them belongs to set A so the probability is $\frac{1}{5}$

5.1 Definition of conditional probability

Suppose that $P(B) > 0$
Then $P(A|B)$ "the probability of A, given B," is
$P(A|B) = P(A \cap B)/P(B)$

Many simply problems in conditional probability can be analyzed by arranging the probabilities into a rectangular array where the rows and columns correspond to mutually exclusive events and each cell gives the probability for the intersection of the "row" and "column" events.

5.2 Example: the "bearded man" problem

At an airport with severe terrorism problems, security personnel have established the following:

- Of male passengers with explosives in their shoes, 60% have beards.
- Of male passengers with no explosives in their shoes, 5% have beards.
- 20% of male passengers have explosives in their shoes.

What is the probability that a bearded male passenger has explosives in his shoes?

Event A is "explosive shoes" while event A' is "non-explosive shoes"
Event B is "bearded" while event B' is "clean-shaven"

¿From the given information we can make a table

	B	B'
A	.12	.08
A'	.04	.76
Sum	.16	84

So $P(A \cap B) = .12, P(B) = .16$, and

$$P(A|B) = P(A \cap B)/P(B) = .12/.16 = 3/4.$$

5.3 The math roommate problem

(an enhanced version of Apostol, p. 488. Example 2)
The Harvard class of 2006 contains $4n$ members who room randomly together in pairs. Precisely $2n$ of them are taking a math course(only one per student). There are three versions of the problem, each with a different answer.

1. In every math course, each student writes the roommate's name on one slip of paper.
2. In every math course, each student writes his or her name on one slip of paper and the roommate's name on another.
3. In all rooms where at least one person is taking a math course the students write each roommate's name on a slip of paper.

From all these slips of paper, one is drawn.
What is the probability that it has the name of a math student on it?"

Solution: Interpret "room randomly" to mean that n rooms have no math students, $2n$ rooms have one math student, n have two math students.
Now count the number of slips with math and non-math names in each case. There are never any slips from rooms with no math students.

1. The $2n$ rooms with one math student each contribute 1 non-math slip. The n rooms with two math students each contribute 2 math slips, because each roommate fills out 1 slip for the roommate.
   So there are $2n$ math slips, $2n$ non-math slips, and the probability of drawing a math slip is 1/2.

2. The $2n$ rooms with one math student each contribute 1 math slip, 1 non-math slip. The n rooms with two math students each contribute 4 math slips, because each roommate fills out 2 slips.
   So there are $6n$ math slips, $2n$ non-math slips, and the probability of drawing a math slip is 3/4.

3. The $2n$ rooms with one math student each contribute 1 math slip, 1 non-math slip. The n rooms with two math students each contribute 2 math slips, because the room (but not each roommate separately) fills out 2 slips.
   So there are $4n$ math slips, $2n$ non-math slips, and the probability of drawing a math slip is 2/3.

Here is the solution in terms of conditional probability. It makes things much easier to distinguish two roommates on the basis of an irrelevant criterion like height rather than talking of "a roommate chosen at random."

1. Given that a student is a math student (event B), what is the probability the the student's roommate is a math student (event A)?
   $P(A \cap B) = 1/4, P(B) = 1/2$; so $P(A|B) = \frac{P(A \cap B)}{P(B)} = 1/2$

2. Given that a student is a math student (event B), what is the probability that the tallest student in the room is a math student (event A)?
   Since 1/4 of all students are math students with a non-math roommate and 1/4 are math students with a math roommate,
   $P(A \cap B) = \frac{1}{4}\frac{1}{2} + \frac{1}{4} = \frac{3}{8}$
   $P(B) = 1/2$ so $P(A|B) = \frac{P(A \cap B)}{P(B)} = 3/4$

3. Given that a room contains at least 1 math student (event B), what is the probability that the tallest student in that room is a math student(event A)?
   Since 1/2 of all rooms have 1 math student and 1/4 of all rooms have 2 math students,
   $$P(A \cap B) = (\frac{1}{2})(\frac{1}{2}) + \frac{1}{4} = \frac{1}{2}$$
   $P(B) = 3/4$ so $P(A|B) = \frac{P(A \cap B)}{P(B)} = 2/3$

5.4 The Monty Hall problem

In a computer game show, there are three doors. Behind one door, chosen at random, is a car C. Behind each of the other two doors is a bicycle. There is one smaller bicycle $B1$ and one larger bicycle $B2$. You select a door. Before revealing whether you have won the car, host Monty Hall, who knows where the car is located, opens a door that you have not chosen and that does not have the car behind it, revealing a bicycle. If he has a choice, he reveals the smaller bicycle. He then asks if you would like to switch your choice to the remaining unopened door.

Suppose (without loss of generality) you pick door 1. There are six ways to arrange the car and the two bicycles behind the three doors

Here are the six 6 equally likely possibilities, along with Monty Hall's action in each case.

Case	Door 1	Door 2	Door 3	Monty opens
1	C	B1	B2	door 2
2	C	B2	B1	door 3
3	B1	C	B2	door 3
4	B2	C	B1	door 3
5	B1	B2	C	door 2
6	B2	B1	C	door 2

Event A is "the car is behind door 1"
Event B is "Monty Hall opened door 2"
So $P(A \cap B) = 1/6$ (case 1 only)
$P(B) = 1/2$ (cases 1, 5, and 6)

So $P(A|B) = P(A \cap B)/P(B) = 1/3$. You still have only 1 chance in 3 of winning the car, and you double your chances of winning by switching to door 3!

A subtle variant: Monty flips a coin to choose a door (not yours) at random and open it. This means that you may discover that you have lost. Now the 6 possibilities are

Car behind 1, heads, opens 2
Car behind 1, tails, opens 3
Car behind 2, heads, opens 2
Car behind 2, tails, opens 3
Car behind 3, heads, opens 2
Car behind 3, tails, opens 3

Suppose that event B is "he opens 2, and there is no car behind it."

Then $P(A \cap B)$ is 1/6 from
Car behind 1, heads, opens 2

$P(B)$ is 2/6 from
Car behind 1, heads, opens 2
Car behind 3, heads, opens 2

Now $P(A|B)$ is 1/2 and there is no advantage to switching.

6 Independent events

Two events A and B are called independent if

$$P(A \cap B) = P(A)P(B)$$

Alternative view: In this case

$$P(A|B) = P(A \cap B)/P(B) = (P(A)P(B))/P(B) = P(A)$$

so that the conditional probability of A is unaffected by knowledge of B.

Simple example:
A = "Drawing a heart"
B = "Drawing an ace"
These are independent — easy argument on Apostol p. 489
A = "Drawing two hearts"
B = "Drawing two aces"
These are not independent since $P(A \cap B) = 0$ while $P(A) > 0$

Slightly more complicated:
A = two fair dice are rolled and the numbers are the same
$P(A) = 1/6$ B = "sum of the numbers on the 2 dice is 8"

There are 5 possibilities: (2,6) (3,5) (4,4) (5,3) (6,2) So $P(A \cap B) = 1/36$ and $P(B) = 5/36$. Thus $P(A|B) = 1/5$ and the events are not independent.

6.1 Three independent events

For three events to be independent it is required that
$P(A \cap B \cap C) = P(A)P(B)P(C)$

Simple but subtle example (modified slightly from Apostol p. 490)

A computer program selects at random from the strings "a," "b," "c," "abc"
Event A: the chosen string contains an "a" $P(A) = 1/2$
Event B: the chosen string contains a "b" $P(B) = 1/2$
Event C: the chosen string contains a "c" $P(C) = 1/2$
Event AB: the chosen string contains an "a" and a "b": $P(AB) = 1/4 = P(A)P(B)$
Event ABC: the chosen string contains an "a" , a "b", and a "c"
$P(ABC) = 1/4$ but $P(A)P(B)P(C) = 1/8$

In other words, once I know that the event $A \cap B$ has occurred, the conditional probability of event C is 1.

7 Compound Experiments

A compound experiment might consist of a die roll followed by a coin flip. In such a case we can construct independent events in a very general manner.

We assume that we know probabilities $P_1(x)$ for the first experiment (the die roll) even if the die is loaded. We assume that we know probabilities $P_2(y)$ for the second experiment (the coin flip) even if the coin is not true.

The outcome of the compound experiment is a pair $\{x, y\}$ e.g. {4, heads}. It is reasonable to define $P(x, y) = P_1(x)P_2(y)$; e.g. 1/12 for each outcome if die and coin are fair.

Note: with this formula, $P(x, y)$ is positive, and the sum over all possible outcomes is 1. The proof is in Apostol.

Now we have to show that the condition for independence is satisfied:

$$P(A \cap B) = P_1(A)P_2(B)$$

This is tricky, since events A, B, and $A \cap B$ all have to be subsets of the same set.

Event A: result of first experiment is in C_1, result of experiment 2 is anything. $P(A) = P_1(C_1)$

Event B: result of second experiment is in C_2, result of experiment 1 is anything. $P(B) = P_2(C_2)$

Event $A \cap B$: result of first experiment is in C_1, result of second experiment is in C_2.

Then from the definition $P(x, y) = P_1(x)P_2(y)$ it follows that $P(A \cap B) = P_1(A)P_2(B)$.

(This obvious-looking result requires a proof – exercise 12 on page 509)

An illustration is in sets.exe, the "Random 4 x 4" probabilities. It works this way:

- Generate 4 random numbers to get $P_1(q)$ for the quotient q. (values 0 - 3)
- Generate 4 other random numbers to get $P_2(r)$ for the quotient r. (values 0 –3)
- Assign to each number $0 <= x <= 15$ $P(x) = P_1(x/4)P_2(x \mod 4)$

Then, no matter what the random numbers, a "remainder event" like $x \mod 4 = 1$ is independent of a "quotient event" like $x <= 7$.

8 Bernoulli trials

Carry out a sequence of identical independent experiments. Select an event S (the same for each experiment) that you call "success." Call its probability p. The complementary event is "failure" F: its probability is $q = 1 - p$.

Theorem: The probability of exactly k successes in n Bernoulli trials is

$$\frac{n!}{k!(n-k)!}p^k q^{n-k}$$

Proof:

The probability of success in k specified trials (for example, the first k) and failure in the remaining $n - k$ trials is

$$p^k q^{n-k}$$

The number of distinct ways of choosing k trials with success is

$$\frac{n!}{k!(n-k)!}$$

The product of the two gives the desired probability of k successes.

Example: What is the probability of getting exactly 2 fives when you roll a die 6 times?

Answer: The probability of getting five on 2 specified rolls, not-5 on the others, is

$$(\frac{1}{6})^2(\frac{5}{6})^4$$

The number of ways of selecting 2 rolls from 6 is $\frac{6\times5}{2} = 15$. So the probability is

$$15 \times \frac{5^4}{6^6}$$

Theorem: Apostol 13.4, with a special case ruled out to save time

If $(n + 1)p$ is not an integer, the most probable number of successes in n Bernoulli trials is $k =$ the greatest integer $< (n + 1)p$.

Let $f(k)$ denote the probability of k successes, and define the ratio $r(k) = f(k)/f(k + 1)$ Note: if $r(k) > 1$, then the probability of one more success is smaller.

Formula for r:

$$r = \frac{k+1}{n-k}\,\frac{1-p}{p}$$

Case 1: $r(0) > 1$. In this case 0 is most probable

Case 2: $r(n - 1) < 1$. In this case n is most probable

Case 3: neither of the above, so we can choose s so that $r(s - 1) < 1$ but $r(s) > 1$. In this case s is most probable.

Now we can think about situations like the following:

Keep rolling a die until a six comes up. What is the most probable number of

rolls?
Now the "universal set" is no longer finite, since there is no maximum number of rolls.

To deal with this situation, we need a new fundamental mathematical idea.

9 Countable and uncountable sets

Sets A and B are in 1-to-1 correspondence if there exists a function $f : A \longrightarrow B$ such that

- The range of f is all of B (f is surjective)
- If x and y are distinct elements of A, $f(x)$ and $f(y)$ are distinct elements of B (f is also surjective, and hence bijective)

Such a function f has an inverse g, and the sets A and B are called equivalent: $A \sim B$ and $B \sim A$

Example: the set of even positive integers is equivalent to the set of all positive integers

A set that is equivalent to $1, 2, ...n$ is a finite set with n elements

A set that is equivalent to 1,2,3... is a countably infinite set.
A computer version of this concept: Imagine a function that produces an infinite sequence of values, all in set B. If it produces each element of B once and only once, then B is countably infinite. If, given any supposed such program, you can invent an element of B that the function can never produce, then B is uncountable.

Properties: proofs left as exercises in Apostol:

a Every subset of a countably infinite set is countable

b The intersection of ANY collection of countable sets is countable

c The union of a COUNTABLE collection of countable sets is countable

d The Cartesian product of a FINITE number of countable sets is countable

Examples:

- The family of 2-element subsets of the positive integers is countable

  Proof: Alternative 1: Arrange the sets in order of increasing larger element, so the computer program produces the bijection
  1: $\{1, 2\}$
  2: $\{1, 3\}$
  3: $\{2, 3\}$

4: $\{1, 4\}$
5: $\{2, 4\}$
and so on. Clearly any 2-element set will appear once and only once

Alternative 2: Arrange the sets in order of increasing sum of elements, so the computer program produces the bijection
1: $\{1, 2\}$
2: $\{1, 3\}$
3: $\{2, 3\}$
4: $\{1, 4\}$
5: $\{2, 3\}$
6: $\{1, 5\}$
7: $\{2, 4\}$
8: $\{1, 6\}$
9: $\{2, 5\}$
10:$\{3, 4\}$
and so on. Clearly any 2-element set will appear once and only once

- The set of rational numbers is countable: Proof: Use the "sum" approach (alternative 2 above) to generate all ordered pairs {numerator, denominator}. Omit any that lead to fractions that are not in lowest terms.
  1: 1/1
  2: 1/2
  3: 2/1
  4: 1/3
  5: 3/1 (note – 2/2 was omitted)
  6: 1/4
  7: 2/3
  8: 3/2
  9: 4/1
  10: 1/5
  11: 5/1 (note: 2/4, 3/3, and 4/2 were all omitted)
  and so on.

- The family of n-element subsets of the positive integers is countable. Proof: Enumerate all n-element sets whose sum is 1, then 2, then 3, then 4, etc.This is a countable union of finite sets.

  The collection of all FINITE subsets of a countable set is countable Proof: This is a countable union of countable sets, one for each value of n.

- The collection of ALL subsets of a countably infinite set is UNcountable Proof (by contradiction): Imagine a computer program that generates a sequence of all subsets of the positive integers: Sometimes the nth element in this sequence will include the integer n; sometime it will not.

Let B be the set of all integers n for which the program generates a subset that does not include n.
Assume that the function generates B as the bth element in the sequence.
Either b is an element of B, or it is not.
Now show that either alternative implies its opposite!

First, assume that b is an element of B.
But the definition of B if that b is not an element of B, so if b is an element of B then b is not an element of B

Second, assume that b is not an element of B. But then b satisfies the criterion for membership in B, so if b is not an element of B then b is an element of B. So either assumption about b leads to a contradiction.
Thus no such b can exist, and B is uncountable.

- The set of real x satisfying $0 < x < 1$ is uncountable

  Proof: Assume the existence of a computer program that list all these numbers sequaentially:

  1: 0.314159...
  2: 0.14285714...
  3: 0.271828....
  and so on.

  Just choose a decimal fraction that differs from #1 in the first digit, from #2 in the second digit, etc. The program can never output that number, hence such a program cannot exist.

10 Probability for countably infinite sets

Imagine a countable collection of disjoint sets. We assume that the union of this (infinite number of) sets is in the Boolean algebra.
Then if

$$P(A_1 + A_2 + A_3 + ...) = P(A_1) + P(A_2) + P(A_3) + ...$$

we say that P is countably additive.

We can specify P by giving its (non-negative) value $P(x)$ for each 1-element subset x. Of course we insist that $P(S) = 1$ for the (countably infinite) universal set S.

Then, since A may contain infinitely many elements.

$P(A) = P(x_1) + P(x_2) + P(x_3) + ...$ may be the sum of an infinite series, guaranteed to converge since its terms are non-negative and the sum cannot exceed 1.

Simple example: Keep rolling a die until you roll a 6. What is the probability that this happens on the nth roll?

Answer: First roll: $P_1 = 1/6$
Second roll: $P_2 = (\frac{5}{6})(\frac{1}{6})$
$n+1$st roll $P_{n+1} = (\frac{5}{6})^n \frac{1}{6}$

The sum of all the probabilities is

$$\frac{1}{6}(1 + \frac{5}{6} + (\frac{5}{6})^2 + (\frac{5}{6})^3 + ...) = \frac{\frac{1}{6}}{1-\frac{5}{6}} = 1$$