MATHEMATICS 152, FALL 2003
METHODS OF DISCRETE MATHEMATICS
Homework Assignment #9
Due: November 25, 2003

Required Problems

1. Apostol, Section 13.4, exercise 1. If no simpler approach occurs to you, try the following:
   To show that sets X and Y are disjoint, show that
   - if x is in X it is not in Y
   - and if x is in Y it is not in X.

   To show that sets X and Y are equal, show that
   - if x is in X it is also in Y
   - and if x is in Y it is also in X.
2. Apostol, Section 13.4, number 4. To prove the formula by induction,
   - first show that it is true for $n = 2$,
   - then assume that it is true for arbitrary n and show that it is true for $n + 1$.
3. Let the universal set S be the set of all 10000 undergraduates at a university. Here are some subsets.
   - A is the set of 2500 freshmen
   - B is the set of 3000 athletes
   - C is the set of 5000 women
   - D is the set of 200 football players

   There are 1200 female athletes, but none of them play football. Of the freshmen, 1000 are athletes and 1250 are women. 400 are both.

   Express each of the following subsets in terms of A, B, and C, and specify its size.

   (a) The set of female upperclassmen.

   (b) The set of male freshman who are not athletes.

   (c) Translate into the language of sets the statement, "If a freshman plays football, then he is a male athlete."

(d) An alumnus asks the university to choose a student at random to receive a scholarship. What is the probability that the recipient is either a male freshman athlete or an upperclass woman?

4. Apostol, section 13.9, exercise 10.

5. Apostol, Section 13.11, #7 and #8. In part c, you need to know that ace, 2, 3, 4, 5 in a suit counts as a straight flush.

6. (This problem is loosely based on some work with a recently-discovered 1774 census of Rhode Island)
   A genealogist, on analyzing names in 18th-century Rhode Island, has ascertained the following:

   - The probability that a male child was a slave was 0.3.
   - Male children who were slaves were given classical names like "Caesar" and "Aesop" with probability 0.6.
   - Male children who were free were given classical names like "Caesar" and "Aesop" with probability 0.2.

   Event A is "the child was a slave" and event B was "the child had a classical name."

   (a) Using set-theoretic notation, express the event "the child was either free or had a classical name" in terms of events A and B and calculate its probability.

   (b) Are events A and B independent? Justify your answer.

   (c) The genealogist encounters the classical name "Cicero Greene". What is the conditional probability, given his name, that Cicero was a slave?

   (d) The genealogist learns that Cicero had one sibling, his brother Roger. Either both were slaves, or both were free. What is the conditional probability that both were slaves?

7. In an eight-team football league there are four referees. At the start of the season coins are distributed to them at random. Three of the coins are fair ones, but the fourth has two heads. Event A is that referee R receives the two-headed coin. An astute coach notices that the first three coin flips of referee R have all come up heads. This is event B. The coach proposes to say to the referee, "Are you using a two-headed coin?" but he wants to know the probability of Event A in order to be sure that he has a good chance of being correct.

   Calculate the conditional probability, given event B, that referee R is using the two-headed coin.

8. (a) Determine how many ways there are to select a subset of 5 of the 13 spades in a deck of cards.You may leave your answer in terms of products of integers, but expand any factorials or binomial coefficients.
   (b) Determine how many distinct bridge hands contain 5 spades, 3 hearts, 3 diamonds, and 2 clubs. As above, you may leave your answer in terms of products of integers.
   (c) Determine how many distinct bridge hands have 5 cards in the longest suit, 2 cards in the shortest suit, and 3 cards in the other two suits.

9. The Queen of Sheba has come to Jerusalem to find a prophet. There are two sorts of prophets: true prophets, who speak the truth nine times out of ten, and false prophets, who speak the truth half the time. Prophet agents are forbidden to reveal explicitly which of their prophets are true ones.

   The queen wants to be more than 90% certain that the prophet she selects is a true one. She hires a prophet agent who brings out three prophets: two true ones and a false one. "2 out of 3 – that's not good enough, is it?" the agent asks. "Sure it is," says the queen, "as long as I can ask one yes-no question." "Ask away," says the agent.

   The queen asks prophet 2, "Is prophet 3 a true prophet?" On hearing the answer she makes her selection and heads home. How did she do it?

   Hint: Event B is "answer is yes", B' is "answer is no." Events A1, A2, A3 are respectively "prophet 1 is a true prophet," etc.

   The queen must be sure that for either answer, her selection (made after hearing the answer) satisfies $P(A|B)$ (or $P(A|B')$) $> .9$.

   For a more fanciful version of this problem, see #3 in the Bayesian Bible.

10. In the admissions office at Monty Hall University there are four interviewers. Three of them, Fl, F2, and F3, are friendly, while the fourth, U, is unfriendly. Every morning the Dean of Admissions assigns them randomly to offices 1, 2, 3, and 4, with an equal probability for each possible assignment. A student arrives for an interview and is asked to select which office he wants to be interviewed in. He chooses office 1 and learns that the interviewer in there is busy for the next half hour. "While you are waiting," says the Dean to the student, "I would like you to meet one of our friendly interviewers. From offices 2, 3, and 4, I will choose the lowest-numbered friendly interviewer." He opens the door of office 2 and introduces the student to an interviewer.
    This is Event B – the lowest-numbered available friendly interviewer was in office 2.
    Event A is that office 1 contains the unfriendly interviewer.

(a) Enumerate all the ways of assigning interviewers to offices that lead to Event B. Assign a probability to each, and show that the sum of these probabilities equals the probability of Event B.

(b) Given that Event B has occurred, determine the conditional probability of event A.

(c) What is the probability that the friendly interviewer to whom the student was introduced by the Dean was F1?

Exploratory Problems

1. Show that the permutations

$$a = (15)(24)(36)$$

and

$$b = (14)(26)(35)$$

generate a subgroup of S_6 that is isomorphic to S_3, but that contains three odd permutations and three even permutations. Hint: think of an equilateral triangle with the vertices numbered 1,2,3 on one side and 4,5,6 on the other.

2. With the preceding problem for inspiration, find a subgroup of $PGL(2, \mathbb{Z}_5)$ that is isomorphic to S_3. It is sufficient to specify one of the four matrices in each coset, but remember that when you "test for equality," being equal up to an overall multiple is good enough. Use the same numbering of lines as in the notes, i.e.

   Line 1: multiples of $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$

   Line 2: multiples of $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$

   Line 3: multiples of $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$

   Line 4: multiples of $\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$

   Line 5: multiples of $\left[\begin{smallmatrix}-2\\1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]$ if you prefer

   Line 6: multiples of $\left[\begin{smallmatrix}-1\\1\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}4\\1\end{smallmatrix}\right]$ if you prefer.

   A permutation specifies the image of line 1 and of line 2, so up to a multiple you know each column of the desired matrix. For the odd permutations you want a trace of zero and a determinant of +2 or -2. For the the even permutations you should get a determinant of +1 or -1 and a trace of +1 or -1. You can save time by letting GL2Z5.exe calculate the effect of the matrices on the lines.

3. For a matrix in $SL(2, \mathbb{Z}_7)$, find a value of the trace that will lead to an element of order 7. It's not sufficient just to show that there are no eigenvalues – use the Cayley-Hamilton theorem to show that A^7 is a multiple of the identity (slightly tedious calculation).

4. Apostol, section 13.4, exercise 10.

5. Apostol, section 13.7, exercise 16.

6. Poker novice Jane picks up her five cards and asks "What did you say the probability was for event A (no two cards of the same rank)?" Veteran Betty tells her. Jane then says "Well, event B (all four suits represented in the hand) has just occurred for me. Is that worth anything?" Betty says, "No, but given that, do you want to know the conditional probability for A, which I'm planning to use when I bet against you?" Calculate $P(A)$, $P(B)$, $P(A \cap B)$, and $P(A|B)$. You'll want a calculator.