MATHEMATICS 152, FALL 2003
METHODS OF DISCRETE MATHEMATICS
Homework Assignment #10
Due: December 9, 2003

Required Problems

If there were an assignment due on December 2, it would include the first five problems.

1. A chance device used by the Lottery Commission can generate any number between 2 and 30 for the "daily numbers game", with the probability of any individual number determined by a secret formula.
   Event A is "the number is prime," and P(A)= 0.4.
   Event B is "the number is less than 15," and P(B) = 0.5.
   Event C is "the number is a prime less than 15," and P(C) = 0.3.

   (a) Are events A and B independent? Explain.

   For each of the following events, specify the event in terms of A and B, and calculate its probability.

   (b) Event D: "the number is a prime greater than or equal to 15."

   (c) Event E: "the number is either prime or less than 15."

   Suppose that today's number has been generated.

   (d) If it is known to be prime, what is the probability that it is also less than 15?

   (e) If it is known to be greater than or equal to 15, what is the probability that it is not prime?

2. Apostol, section 13.14, exercise 14.

3. At Awesome State University, grade inflation has become so extreme that the NCAA requires athletes to get A's in at least half their courses to remain eligible. A star football player can choose to take four courses ($p = 1/2$ for an A) or just three courses ($p = 2/3$ for an A). Which choice gives the higher probability of remaining eligible?

4. You have to deliver crucial supplies using airplanes with very unreliable engines. Each engine has a probability p of lasting for the entire flight, and engine failures are independent events. If half or more of the engines fail, the plane crashes. Your choice is between using two-engine planes, which crash if either engine fails, or 4-engine planes, which crash if two or more engines fail.

(a) What is the probability that exactly three engines on a four-engine plane will survive?

(b) Determine for what value of p the probability that a plane will not crash is the same for 2-engine and 4-engine planes.

(c) For this value of p, would 6-engine planes be a better choice?

5. Boxcar Bob owns three dice. Two of them are unloaded, but the third has $p = 1/2$ for a 6, and $p = 1/10$ for the numbers 1 through 5. Event B is that he rolls a randomly chosen die three times, and a 6 appears precisely once. Event A is that Bob is using the loaded die.

   (a) Calculate $P(A \cap B)$, $P(A' \cap B)$, and $P(B)$.

   (b) Given event B, what is the conditional probability that Bob is using the loaded die?

6. Apostol, section 13.18, exercise 14.

7. Apostol, page 508, section 13.23, exercise 8.

8. The Afghan warlord General Dostum has captured a female storyteller named Bernoulli (like most Afghans, she uses only one name). She offers to tell him a story every day, on condition that he will not turn her over to the CIA that day if her story makes him laugh. Since Dostum has a fine sense of humor, the probability that a story will NOT make him laugh is $p = 1/5$.

   (a) What is the probability that she is turned over to the CIA after telling precisely 3 stories?

   (b) What is the probability that she is turned over to the CIA on or before the third day?

   (c) Donald Rumsfeld is asked by a reporter, on the day of her capture, "What day is the CIA most likely to gain custody of Bernoulli?" What is his correct answer? Explain.

9. Ill-tempered pooch Fido greets the postal delivery person with a growl every day, but he only bites him or her with probability p. In Fido's town, the legal maxim "every dog gets one bite" is honored, but the second time that Fido bites, it is off to the pound with him.

   Find a formula for the probability P_n that Fido delivers his second and final bite on day n. (Hint: the first bite could have been delivered on any of the $n-1$ previous days). By considering the binomial expansion of $(1-q)^{-2}$, show that

$$\sum_{n=2}^{\infty} P_n = 1$$

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10. Apostol, section 13.20, exercises 9a, 9c, and 10b. For the second of these, do a proof by contradiction, and use the facts that every interval of positive length contains at least one rational number and that the rational numbers are countable

11. The U.S. Supreme Court, in a unanimous decision, has decreed that all future high-school graduates will be given serial numbers and that every university must immediately submit an acceptance list of the serial numbers of all the students they will ever admit. The Harvard Admissions Committee convenes for one last time to agree on its acceptance list. An amateur programmer on the committee demonstrates a program P he has written, which produces output that starts like this:

    Possible acceptance lists for Harvard College – choose one
    1: 2, 3, 5, 8, 13, 21, 34, ...
    2: all even-numbered students
    3: all students with prime serial numbers
    4: ....

    "All we have to do is select one of the lists that my program can produce," he announces, "and our job is done forever. My program won't waste your time, since every possible list will be produced once and only once."

    "Wait a minute," says a mathematician on the committee. "I have another idea for an acceptance list – call it list A. Since your list 1 rejects student 1, my list A will admit him. Since your list 2 admits student 2, list A will reject him. In general, for student k, we will run your program and see if your list k has student k on it. If not, my list A will accept him."

    "I like the idea of your list A," says a sociologist, "since it counteracts any discrimination that may unwittingly be built into the computer program. But if we just run the program for a while, it should print out list A, say as list number n. Then we just tell the Dean that we've chosen list number n and we're done. I'm curious to know whether student number n will be accepted or not."

    (a) On the basis of what you know about countability and one-to-one correspondence, explain whether it is possible for program P to output all possible acceptance lists.

    (b) Prove your answer to part a by giving a careful (paradoxical) answer to the sociologist's question about whether student number n is accepted according to list number n.

Exploratory Problems

1. The last problem from the "Bayesian Bible" that was attached to outline 3.

2. (At Marcia Weis's invitation, I once presented the solution to this problem at a bridge lecture on board the Grand Princess.)
   The year is 2202, and Earth can support its large population only by having billions of people on cruise ships. Marcia Weis, the bridge director for Megaprincess Lines, is preparing a duplicate bridge tournament for about 10 million players. In the opening deal, she arranges for the North-South pairs all to have the same hands, including 9 spades – all but the queeen, 4, 3, and 2. She then distributes the remaining 26 cards (4 spades and 22 others) in every possible way between East and West. Thus every declarer, in trying to guess how the missing cards are divided between East and West, will know that each possible outcome will occur precisely once on a cruise somewhere.

   In what follows, let $N = \frac{22!}{(13!11!)} = 4522$ and express all answers as a multiple of N.

   (a) How many different East hands of 13 cards can Marcia prepare?

   (b) In how many of these hands does East have all four of the missing spades?

   (c) In how many of these hands does East have all of the missing spades except the queen?

   (d) In how many of these hands does East have precisely three of the missing spades?

   (e) In how many of these hands does East have precisely two of the missing spades? (As a check, add together this number, twice the answer to part b, and twice the answer to part d. This should agree with the answer to part a).

   (f) Since Princess passengers have been taught the maxim 'Eight ever, nine never" at the mandatory lifeboat drill, they will all play this deal the same way, by leading the ace and king of spades and hoping that East or West will be forced to play the queen. This approach will succeed if East and West each has two spades or if either East or West has only the queen while the other has the remaining three spades. In how many of the hands will the approach succeed? What is the probability of success?

   (g) Suppose that you are North, you lead the ace of spades, and both East and West play their lowest spade, which is not the queen. What is the conditional probability that East and West now each has one of the missing spades? What is the conditional probability that East has both of the missing spades?

3. Apostol, Section 13.23, page 509, exercise 12.

4. Apostol, Section 13.20, exercises 7 and 8.

5. Legendary baseball manager Bill Veeck once sent a midget to the plate in hopes that he could draw a walk. The midget had instructions to take pitches until he walked or struck out, and of course because of his small strike zone, the probability p that the pitcher would throw a ball was fairly high. Calculate, as a function of p, the probability that the midget will walk: that he will receive 4 balls before 3 strikes. Although there is no closed-form solution to this problem, you can get the answer as a sum of three terms in two different ways, and both should lead to the same polynomial in p.

   (a) Add together the probability for the midget to walk on 4, 5, or 6 pitches, taking 0, 1, or 2 strikes before the final ball.

   (b) Assume that the pitcher throws 6 pitches, at which point the midget has struck out if he has not walked. Add together the probability for 4, 5, or 6 of them to be balls.

   If you are after Grady Little's job, you may want to evaluate the answer for a few values of p, but you are not required to. If you do, consider using Mathematica.