Worksheet: knot coloring

Math 139

This worksheet is to be done in class Monday 12th February. Gerardo will outline the theory and guide you through the exercises. The answers to the exercises will be compiled and put on the class website. You will get credit towards your HW grade from participating in class.

Reference: Charles Livingston "Knot Theory" Chapter 3 sections 2 through 4.

Definition 1 *A knot diagram is called* colorable *if each arc can be drawn using one of three colors such that 1) at least two of the colors are used and 2) at any crossing at which two colors appear, all three appear.*

Theorem 1 *If a diagram of a knot K is colorable, then every diagram of K is colorable.*

Definition 2 *A knot is called colorable if its diagrams are colorable.*

1. Which of the knot diagrams (in the standard table of knots) with seven or fewer crossings are colorable?

2. For which integers n is the $(2, n)$-torus knot colorable? For which values of n is the n-twisted double of the unknot colorable? (Gerardo will draw the n-twisted double of the unknot, where $2n$ is the number of crossings in the vertical band.)

3. What can you say about the colorability of the (p, q, r)-pretzel knots?

4. The definition of colorability is often stated slightly differently. The requirement that at least two colors are used is replaced with the condition that all three colors appear.

(a) Show that the unlink of two components has a diagram which is colorable using all three colors and another diagram which is colorable with exactly two colors.

(b) Why is it true that for a knot, once two colors appear all three must be used, whereas the same statement fails for links?

(c) Prove that the Whitehead link is nontrivial, by arguing that it is not colorable.

Definition 3 *A knot diagram can be labeled mod p if each edge can be labeled with an integer from 0 to $p-1$ such that 1) at each crossing the relation $2x - y - z = 0$ mod p holds, where x is the label on the overcrossing and y and z the other two labels, and 2) at least 2 labels are distinct. (Here p is prime.)*

5. Determine which knots with 6 or fewer crossing can be labeled mod 5.

6. For what primes p can the trefoil knot diagram be labeled mod p?

7. (a) Show that if all the labels of knot that is labeled 3 are multiplied by 5, the resulting labeling is a labeling mod 15.
(b) Show that no knot can be labeled mod 2.

Matrices: We can simplify the problem of determining mod p colorings by using linear algebra. Label each arc of the diagram with a variable, say x_i. At each crossing a relation between the variables is defined: if arc x_i crosses over x_j and x_k, then $2x_i - x_j - x_k = 0$ mod p. A knot can be labeled mod p if there is a mod p solution to this system of equations with not all x_i equal. Also note we have the condition that the solution has at least two of the x_i distinct.

Theorem 2 *There is an $n \times n$ matrix corresponding to a knot diagram with n arcs. Deleting any one column and any one row yields a new matrix. The knot can be labeled mod p if and only if the corresponding set of equations has a nontrivial mod p solution.*

Definition 4 *The* determinant *of a knot is the absolute value of the determinant of the associated $(n-1) \times (n-1)$ matrix constructed above. The* mod p rank *of a knot is the mod p nullity of the associated $(n-1) \times (n-1)$ matrix.*

Theorem 3 *The determinant of a knot and its mod p rank are independent of the choice of diagram and labeling.*

8. For each knot with 6 or fewer crossings, find the associated matrix, and its determinant. In each case, for what p is there a mod p labeling?

9. The knots 8_{18} and 9_{24} both have determinant 45. Check that one has mod 3 rank 1, while the other has mod 3 rank 2. The knots 8_8 and 9_{45} both have determinant 25. Compute their mod 5 ranks.

Final projects:

• The determinant and mod p ranks are captured by stronger invariants called *torsion invariants.*

• There is a way to extend the use of matrices and determinants further and define the Alexander polynomial.