Symplectic Geometry Lecture 11.

Symplectic group actions and moment maps.

Background on Lie groups.

A Lie group is a group G with a manifold structure on G such that group multiplication is a smooth map of $G \times G \to G$ and the inverse map $g \mapsto g^{-1}$ of $G \to G$ is a smooth map. A Lie subgroup $H \subset G$ is a subgroup which is also a submanifold. By a theorem of Cartan every closed subgroup of a Lie group is an embedded Lie subgroup i.e. smoothness is automatic. In this case the homogeneous space G/H inherits a unique manifold structure such that the quotient map is smooth.

Left and right translation.

For $g \in G$, let

$$L_g : G \to G$$

denote the map

$$a \mapsto ga.$$

We call this map left translation by g. Similarly,

$$R_g : G \to G, \quad a \mapsto ag$$

denotes right translation by g.

A vector field X on G is called **left invariant** if

$$(dL_g)_a(X(a)) = X(ga)$$

for all $g, a \in G$. A similar definition for right invariant vector fields.

f related vector fields.

The following notation is useful: Let $f : P \longrightarrow Q$ be a smooth map between manifolds. Let X be a vector field on P and Y be a vector field on Q. We say that these vector fields are f-**related** if for every $p \in P$ we have

$$df_p(X(p)) = Y(f(p)).$$

For example, in the special case that f is a diffeomorphism, being f related is the same as saying that

$$Y = (f^{-1})^* X.$$

So a vector field on a Lie group G is left invariant if and only if it is L_g related to itself for all $g \in G$.

Left invariant vector fields generate ***right*** translations.

Suppose that X is a left invariant vector field and that ϕ_t is the flow that it generates. Then $L_g \circ \phi_t = \phi_t \circ L_g$ for all g so that ϕ_t must consist of *right* translations.

The Lie algebra of a group.

Let $\mathfrak{X}^L$ denote the space of left invariant vector fields. If $X, Y \in \mathfrak{X}^L$ then $[X, Y] \in \mathfrak{X}^L$ by the definition of left invariance. But any $X \in \mathfrak{X}^L$ is completely determined by its value $X(e)$ at the identity. So as a vector space,

$$\mathfrak{X}^L \cong \mathfrak{g} := T_e G.$$

The space $\mathfrak{g} = T_e G$ with bracket induced from $\mathfrak{X}^L$ becomes a Lie algebra called **the Lie algebra of** G. That is, we define the Lie bracket on $\mathfrak{g}$ by the formula

$$[A, B] = [X^L, Y^L](e)$$

where X^L is the left invariant vector field with $X^L(e) = A \in \mathfrak{g}$ and Y^L is the left invartiant vector field with $Y^L(e) = B \in \mathfrak{g}$.

The case of matrix groups.

For matrix Lie groups, i.e. closed subgroups of $Gl(n, \mathbb{R})$, this definition of Lie bracket coincides with the commutator of matrices.

The exponential map.

Let X be the left invariant vector field with $X(e) = A \in \mathfrak{g}$. As mentioned, the flow it generates consists of a one parameter group of right multiplications. We denote this one parameter group by $t \mapsto R_{\exp tA}$. In particular, the trajectory of X which passes through the identity e at $t = 0$ is the one parameter subgroup $t \mapsto \exp tA$ of G. Setting $t = 1$ we get a map

$$\exp : \mathfrak{g} \to G, \qquad A \mapsto \exp A.$$

For matrix Lie groups exp is the usual exponential of matrices.

Conjugation and the adjoint representation.

For $g \in G$, the map

$$A_g : G \to G, \quad A_g := L_g \circ R_{g^{-1}}$$

is just conjugation by g:

$$A_g(a) = gag^{-1}.$$

In particular, $A_g(e) = e$. So the differential of A_g at e is a linear map of $\mathfrak{g} \to \mathfrak{g}$ which we denote by Ad_g:

$$\mathrm{Ad}_g := d(A_g)_e : \quad \mathfrak{g} \to \mathfrak{g}.$$

Group actions.

Let G be a Lie group,

Definition 1 *An action of G on a manifold Q is a smooth map*

$$\mathcal{A} : G \times Q \to Q, \quad (g, q) \mapsto \mathcal{A}(g, q) =: \mathcal{A}_g(q)$$

such that the map $g \mapsto \mathcal{A}_g$ is a homomorphism of $G \to$ Diff (Q).

When there is no risk of confusion we will write gq instead of $\mathcal{A}(g, q)$. Then the condition that $g \mapsto \mathcal{A}_g$ be a homomorphism takes the form of an "associtive law":

$$a(bq) = (ab)q.$$

Actions of a group on itself.

For example, there are three natural actions of any Lie group G on itself: The left action $L_g : a \mapsto ga$, the right action $R_g : a \mapsto ag^{-1}$ and the conjugation action $A_g : a \mapsto gag^{-1}$.

G-manifolds and equivariant maps.

A manifold Q together with an action of G on Q is called a G-manifold. A map

$$F : Q_1 \to Q_2$$

between two G manifolds is called G-**equivariant** if it intertwines the G actions; that is if

$$gF(q) = F(gq)$$

for all $g \in G$ and $q \in Q_1$.

Action of a Lie algebra.

Let $\mathfrak{g}$ be a Lie algebra.

Definition 2 *An action of* $\mathfrak{g}$ *on a manifold* Q *is a smooth vector bundle map*

$$\mathfrak{g} \times Q \to TQ, \quad (A, q) \mapsto A_Q(q)$$

such that the map

$$\mathfrak{g} \to \mathfrak{X}(Q). \quad A \mapsto A_Q$$

is a Lie algebra homomorphism:

$$[A, B]_Q = [A_Q, B_Q]$$

where the bracket on the left is the bracket of the Lie algebra $\mathfrak{g}$ *and the bracket on the right is the Lie bracket of vector fields on* Q.

Generating vector fields for a group action.

Definition 3 *For $A \in \mathfrak{g}$ the* **generating vector field** A_Q *on Q is the vector field which generates the flow*

$$q \mapsto \exp(-tA)q$$

on Q.

Notice the minus sign. There are various conventions; I am following the convention used by Meinrecken. The reason for the minus sign is: If G acts on Q, then to get a (left) action on functions on Q, we must define af as

$$(af)(x) = f(a^{-1}x).$$

Conventions

$$(af)(x) = f(a^{-1}x).$$

So the action of the one parameter group $\exp tA$ on functions is $f \mapsto (\exp tA)f$ where

$$((\exp tA)f)(x) = f((\exp -tA)x).$$

We will soon see that this gives a (Lie algebra) action of $\mathfrak{g}$ on Q. In the Hamiltonian case this will guarantee that we get a homomorphism of our Lie algebra into the Lie algebra of Poisson brackets as we shall soon see. However this convention leads to some mildly unpleasant minus signs in some examples.

For example, consider the three natural actions of a Lie group on itself. For $A \in \mathfrak{g} = T_eG$, let A^L denote left invariant vector field with $A^L(e) = A$. Similarly, let A^R be the right invariant vector field with $A^R(e) = A$. The generating vector field for the left action is right invariant since the left action commutes with the right action and its value at e is $-A$. So the generating vector field for the left action is $-A^R$. Similarly the right action is generated by A^L and the conjugation action by $A^L - A^R$.

A group action determines an action of its Lie algebra.

Proposition 1 *For any action of G on a manifold Q the map $B \mapsto B_Q, B \in \mathfrak{g}$ is an action of the Lie algebra $\mathfrak{g}$ of G on Q. Furthermore, for every $g \in G$ and $q \in Q$ and $B \in \mathfrak{g}$*

$$d\left(\mathcal{A}_g\right)_q (B_Q)(q) = (\mathrm{Ad}_g\, B)_Q(gq).$$

In other words the vector fields B_Q and $(Ad_g B)_Q$ are $\mathcal{A}_g$-related. In still other words, this says that

$$g^* B_Q = (\mathrm{Ad}_{g^{-1}}\, B)_Q. \tag{2}$$

Proof by "universality".

Proof. Let $P = G \times Q$ with $g \in G$ acting on P via

$$(a, q) \mapsto (ag^{-1}, q).$$

In other words, G acts on the G component by the right action, and doesn't act at all on the Q component. So under the identification $TP = TG \times TQ$ the generating vector fields for this action are exactly

$$(B^L, 0).$$

Since we defined the Lie bracket on $\mathfrak{g}$ as the Lie bracket of these left invariant vector fields we know the first part of the Proposition for this case.

Recall the three natural actions of any Lie group G on itself: The left action $L_g : a \mapsto ga$, the right action $R_g : a \mapsto ag^{-1}$ and the conjugation action $A_g : a \mapsto gag^{-1}$.

For any $B \in \mathfrak{g}$ we have $dR_g B^L = dA_g B^L$ at any point of G since $A_g = R_g \circ L_g$ and B^L is left invariant. So we know the second part as well for the case of P.

Consider the map

$$F : P \to Q, \quad F(a, q) := a^{-1}q.$$

We have

$$F((ag^{-1}, q)) = (ag^{-1})^{-1}q = ga^{-1}q = gF((a, q)).$$

In other words, the map F is G-equivariant. Then since the Proposition is true for P and the vector fields B_P and B_Q are F-related, the Proposition is true for Q. $\square$

The adjoint representation again.

Any action of G on Q gives rise to an action on TQ such that the projection $TQ \to Q$ is G-equivariant and also to an action of G on T^*Q such that the projection $T^*Q \to Q$ is G-equivariant. In particular we can apply this to the conjugation action of G on itself. Since the conjugation action fixes the identity, we get the adjoint action of G on $\mathfrak{g}$ as we have seen, and also the co-adjoint action of G on $\mathfrak{g}^*$. The two actions are related by

$$\langle g \cdot \ell, A \rangle = \langle \ell, \mathrm{Ad}_{g^{-1}} A \rangle. \tag{3}$$

Symplectic actions.

Let (M, ω) be a symplectic manifold. A G-action $g \mapsto \mathcal{A}_g$ on M is called **symplectic** if $\mathcal{A}_g \in \mathrm{Symp}(M)$ for all $g \in G$. In other words, if

$$\mathcal{A}_g^* \omega = \omega \quad \forall\ g \in G.$$

Similarly, an action of a Lie algebra $\mathfrak{g}$ is called symplectic if $A_M \in \mathfrak{X}(M, \omega)$ for all $A \in \mathfrak{g}$. Clearly, if $\mathfrak{g}$ is the Lie algebra of G, then the $\mathfrak{g}$ action defined by a symplectic G action is symplectic.

Weakly Hamiltonian actions.

Recall that if F is a smooth function on a symplectic manifold (M, ω) then the vector field X_F denotes the (unique) vector field which satisfies

$$i(X_F)\omega = dF.$$

A symplectic G-action or a $\mathfrak{g}$-action is called **weakly Hamiltonian** if all the vector fields A_M are Hamiltonian. In other words, if for each $A \in \mathfrak{g}$ there is a smooth function $\Phi(A)$ on M such that

$$A_M = X_{\Phi(A)}.$$

One can always choose $\Phi(A)$ to depend linearly on A, by fixing the values of $\Phi(A_i)$ for the A_i in a basis of $\mathfrak{g}$ and then extending linearly.

The weak moment map.

One can always choose $\Phi(A)$ to depend linearly on A, by fixing the values of $\Phi(A_i)$ for the A_i in a basis of $\mathfrak{g}$ and then extending linearly. Then the map

$$A \mapsto \Phi(A)$$

can be viewed as a $\mathfrak{g}^*$ valued function on M:

$$\Phi \in C^\infty(M) \otimes \mathfrak{g}^*.$$

In other words, a symplectic G-action on a symplectic manifold (M, ω) is weakly Hamiltonian if there is a smooth map, called the **moment map**

$$\Phi : M \to \mathfrak{g}^*$$

i.e. $\Phi \in C^\infty(M) \otimes \mathfrak{g}^*$ such that

$$i(A_M)\omega = d\langle \Phi, A \rangle \quad \forall\ A \in \mathfrak{g}. \tag{4}$$

Hamiltonian actions.

Definition 4 *A weakly Hamiltonian G-action is called* **Hamiltonian** *with moment map* Φ *if the moment map can be (and has been) chosen so as to be an equivariant map from* M *to* $\mathfrak{g}^*$ *relative to the co-adjoint action of* G *on* $\mathfrak{g}^*$.

Similarly one defnes moment maps for Hamiltonian $\mathfrak{g}$ actions: one requires Φ to be $\mathfrak{g}$ equivariant in this case.

The induced action of a subgroup.

It is obvious that if we are given a Hamiltonian action of G on M and if H is a subgroup of G then the induced action of H is Hamiltonian with moment map the composition of the projection $\mathfrak{g}^* \to \mathfrak{h}^*$ dual to the injection $\mathfrak{h} \to \mathfrak{g}$ with the moment map $\Phi : M \to \mathfrak{g}^*$.

More generally, if $f : H \to G$ is a smooth homomorphism from a Lie group H to G, it induces a Lie algebra homomorphism $\mathfrak{h} \to \mathfrak{g}$ and hence a dual linear map $\mathfrak{g}^* \to \mathfrak{h}^*$ whose composition with Φ gives a moment map for the induced action of H on M.

The equivariance condition.

Let us write out the equivariance condition on the moment map in more detail: It says that $(\mathrm{Ad}^*)(g) \circ \Phi = \Phi \circ \mathcal{A}_g$ or

$$\langle \Phi(gm), B \rangle = \langle \Phi(m), Ad_g^{-1} B \rangle$$

for all $m \in M, g \in G, B \in \mathfrak{g}$. If we set $g = \exp -tA$ and take the derivative at $t = 0$ this becomes

$$\langle A_M \Phi(m), B \rangle = \langle \Phi(m), [A, B] \rangle$$

or simply

$$\langle A_M \Phi, B \rangle = \langle \Phi, [A, B] \rangle \qquad (5)$$

as $\mathfrak{g}^*$ valued functions on M.

Homomorphism to Poisson brackets.

Remember that $C^\infty(M)$ is a Lie algebra under Poisson brackets.

Lemma 1 *For a Hamiltonian G-action on M the map*

$$\mathfrak{g} \to C^\infty(M), \quad A \mapsto \langle \Phi, A \rangle$$

is a homomorphism of Lie algebras,

Proof. Let us write Φ^A for $\langle \Phi, A \rangle$. Then by(5)

$$\{\Phi^A, \Phi^B\} = X_{\Phi^A}\Phi^B = A_M \Phi^B = \Phi^{[A,B]}. \quad \square$$

Modifying the weak moment map.

If we are given a weakly Hamiltonian action of a compact Lie group G we can always modify the moment map so that it becomes equivariant. Indeed, if Ψ is a (possibly non-equivariant) moment map, define

$$g \bullet \Psi = Ad^*_{g^{-1}} \circ \Psi \circ \mathcal{A}_{g^{-1}}.$$

So a weak moment map Φ is equivariant if and only if

$$g \bullet \Phi = \Phi, \qquad \forall\; g \in G.$$

Let Ψ be a (possibly non-equivariant) moment map. I claim that $g \bullet \Psi$ is again a moment map. Indeed, for any $A \in \mathfrak{g}$ we have

$$
\begin{aligned}
d\langle g \bullet \Psi, A\rangle &= (g^{-1})^* d\langle \Psi, \mathrm{Ad}_{g^{-1}} A\rangle \\
&= (g^{-1})^* i((\mathrm{Ad}_{g^{-1}})_M \omega \\
&= (g^{-1})^* i(g^*(A_M))\omega \quad \text{by (2)} \\
&= (g^{-1})^* i(g^* A_M) g^* \omega \\
&= (g^{-1})^* (g^*(i(A_M)\omega)) \\
&= i(A_M)\omega. \quad \square
\end{aligned}
$$

If we then average the $g \bullet \Psi$ over the group G we get an equivariant moment map.

Another case in which we can modify a a (possibly non-equivariant) moment map to make it equivariant is the case where M is compact. In this case we can always subtract a constant element of $\mathfrak{g}^*$ so that

$$\int_M g \bullet \Phi dm = 0$$

where we are integrating relative to the measure associated to the Liouville form. This then fixes the arbitrary constant in Φ and we have

$$\int_M g \bullet \Phi dm = \int_M \Phi dm = 0 \quad \forall g \in M$$

since g preserves the Liouville form.

Moment maps for exact symplectic manifolds.

Suppose that (M, ω) is exact with $\omega = -d\alpha$ and that our group action preserves α. Then

$$D_{A_M}\alpha = 0 \quad \forall\ A \in \mathfrak{g}^*$$

and by Weil's formula

$$i(A_M)d\theta + di(A_M)\theta$$

which says that if we define $\Phi : M \to \mathfrak{g}^*$ by

$$\langle \Phi, A \rangle := \langle \alpha, A_M \rangle \tag{6}$$

then Φ is a weak moment map.

To check equivariance observe that

$$\langle g^*\Phi, A\rangle = g^*\langle \Phi, A\rangle = g^*\langle \alpha, A_M\rangle$$

$$= \langle g^*\alpha, g^*A_M\rangle = \langle \alpha, g^*A_M\rangle = \langle \alpha, (Ad_{g^{-1}}A)_M\rangle.$$

Applied to g^{-1} this says that $g \bullet \Phi = \Phi$. $\square$

Induced cotangent actions.

$$\langle \Phi, A \rangle := \langle \alpha, A_M \rangle \tag{6}$$

For example, if we are given a G-action on manifold Q we get a Hamiltonian G-action on the cotangent bundle T^*Q where now $\alpha = \alpha_Q$ is the fundamental one form of the cotangent bundle. The definition (6) now reads

$$\langle \Phi, A \rangle (q, \xi) = \langle \xi, A_Q(q) \rangle \quad \xi \in T^*_q(Q),\ A \in \mathfrak{g}. \tag{7}$$

Total linear momentum and its conservation.

We will spend a lot of time studying the actions of G on T^*G. But let us first study the case where $G = V$ is a vector space which acts on itself by right translation. So the action of $w \in V$ sends $v \in V$ to $v - w$ and the one parameter group $\exp -tu$, $u \in V$ sends

$$v \mapsto v + tu.$$

So if we identify $\mathfrak{g} = T_0V$ with V, the vector field u_V associated to $u \in \mathfrak{g} \cong V$ is just the constant vector u:

$$u_V(x) = u \in T_xV$$

under the identification of T_xV with V. So (7) says that

$$\langle \Phi(x, \xi), u \rangle = \langle \xi, u \rangle.$$

Even more succinctly if we identify $T_x^* V$ with V^*, this says that

$$\Phi(x, \xi) \equiv \xi.$$

So far we have taken $G = V$. Suppose that V is a direct sum of several copies of the same vector space W

$$V = W \oplus W \oplus \cdots \oplus W \quad n \text{ copies}$$

and consider the subgroup H of $G = V$ consisting of the diagonal elements:

$$H := \{(w, w, \ldots, w)\}.$$

If we identify the Lie algebra of $H = W$ with W, this gives an injection of Lie algebras, and the dual projection $V^* \to W^*$ is

$$\xi = (\mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^n) \mapsto \mathbf{p}^1 + \mathbf{p}^2 + \cdots + \mathbf{p}^n.$$

We call the expression $\mathbf{p}^1 + \mathbf{p}^2 + \cdots + \mathbf{p}^n$ the **total linear momentum** (relative to the right diagonal action of W on V).

For example, suppose that we have a Hamiltonian function $\mathcal{H}$ on $M = T^*V$ such that $\mathcal{H}$ is invariant under the induced action of W on T^*V. (Think of the case where $W = \mathbb{R}^3$ so that M is the phase space for n particles moving in $\mathbb{R}^3$ and the Hamiltonian is invariant under simultaneous translation of all the particles.) This says $X_{w_M}\mathcal{H} = 0$ for all $w \in W$. We know that this implies that

$$D_{\mathcal{H}}(\mathbf{p}^1 + \mathbf{p}^2 + \cdots + \mathbf{p}^n) = 0.$$

In other words, the function $\mathbf{p}^1 + \mathbf{p}^2 + \cdots + \mathbf{p}^n$ is constant on the trajectories of $\mathcal{H}$. This is the law of **conservation of total linear momentum**.

Gl(V) acting on *V*.

We stick with the case that $Q = V$ is a vector space, but take $G = Gl(V)$ the group of linear transformations of V. If we choose a basis of V then we can identify $Gl(V)$ with $Gl(d, \mathbb{R})$, the group of invertible $d \times d$ matrices where $d = \dim V$. Let us choose a basis. Then $\mathfrak{g}$ becomes identified with the space of all $d \times d$ matrices with Lie bracket the commutation of matrices and with exponential map the exponential map for matrices. We compute the generating vector field $A_V = A_{\mathbb{R}^d}$: If f is a smooth function on $V = \mathbb{R}^d$ then

$$(A_V f)(q) = \frac{d}{dt} f\left((\exp -tA)q)\right)_{|t=0} = -\sum_j \frac{\partial f}{\partial q_j}(Aq)_j$$

$$= -\sum_{jk} A_{jk} q_k \frac{\partial f}{\partial q_j}.$$

$$(A_V f)(q) = \frac{d}{dt} f\left((\exp -tA)q)\right)_{|t=0} = -\sum_j \frac{\partial f}{\partial q_j}(Aq)_j$$

$$= -\sum_{jk} A_{jk} q_k \frac{\partial f}{\partial q_j}.$$

So

$$A_V = -\sum_{jk} A_{jk} q_k \frac{\partial}{\partial q_j}.$$

If $p_1, \ldots p_d$ are the coordinates dual to the q's, we see from (7) that

$$\langle \Phi, A \rangle = -\sum_{jk} A_{jk} p_j q_k.$$

Angular momentum.

$$\langle \Phi, A \rangle = -\sum_{jk} A_{jk} p_j q_k.$$

Suppose V has a scalar product and we restrict to the orthogonal group and orthogonal Lie algebra. If our basis is orthonormal, then the corresponding matrices A are anti-symmetric, and so we can right the previous sum as

$$\frac{1}{2} \sum_{jk} A_{jk} (p_j q_k - p_k q_j).$$

If we introduce the matrix valued function on $T^*V = V \oplus V^*$

$$B(q,p) := (B_{k\ell}(q,p)) \qquad B_{k\ell}(q,p) = p_k q_\ell - p_\ell q_p$$

then the moment map becomes

$$\langle \Phi, A \rangle = -\frac{1}{2} trAB, \qquad B = B(q,p).$$

Total angular momentum.

$$\langle \Phi, A \rangle = -\frac{1}{2} tr AB, \quad B = B(q,p).$$

The matrix valued function B (up to some scalar factors) is called the angular momentum. Once again, if V is an orthogonal direct sum of copies of the same space W (now with a scalar product) and we restrict to the "diagonal subgroup" the corresponding moment map is the total angular momentum. This is a conseved quantity under the flow generated by a Hamiltonian $\mathcal{H}$ if $\mathcal{H}$ is conserved under "simultaneous" rotations corresponding to the diagonal subgroup of $D(V)$.