Symplectic Geometry Lecture 10

Reduction

Differential systems aka distributions.

Let M be a manifold and $\mathfrak{D}$ a sub-bundle of TM. Sometime $\mathfrak{D}$ is called a differential system and is sometimes called a distribution. For example, if α a nowhere vanishing one form on M, then at each $x \in M$ we get a differential system of codimension one where $\mathfrak{D}_x \subset T_xM$ consists of those $v \in T_xM$ such that $\langle \alpha_x, v \rangle = 0$.

A non-integrable distribution.

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If there are functions (locally defined) T and S such that $\alpha = TdS$ then at each x_0 the space $\mathfrak{D}_{x_0}$ is tangent to the hypersurface $S = S(x_0)$. But if $\alpha = TdS$ then $d\alpha = dT \wedge dS$ and so

$$\alpha \wedge d\alpha \equiv 0.$$

On other hand, consider the form $\alpha = dz + xdy$ (say on three dimensional space). Then

$$\alpha \wedge d\alpha = dx \wedge dy \wedge dz$$

is nowhere zero. So we can not write $\alpha = TdS$ on any neighborhood. The planes of the differential system $\mathfrak{D}$ defined by α do not "fit together" to be tangent to surfaces.

Foliations, submersions, and fibrations.

Submersions.

Let Q and B be manifolds. A **submersion** is a smooth map $f: Q \to B$ such that for all $q \in Q$ the differential

$$df_q : T_qQ \to T_{f(q)}B$$

is surjective. For any submersion the "fibers"

$$f^{-1}(a). \quad a \in B$$

are smooth embedded submanifolds of Q of dimension $k = n - \dim B$. This follows from the implicit function theorem.

Submersions are foliated.

In fact Q is **foliated** by such submanifolds in the following sense: Every point $q \in Q$ has a coordinate neighborhood U with coordinates $x_1, \ldots, x_n$ with q corresponding to $x = 0$, and $f(q)$ a coordinate neighborhood with coordinates $y_1, \ldots y_{n-k}$ with $f(q)$ corresponding to $y = 0$ such that in terms of these coordinates, the map f is given by by projection to the first $n-k$ coordinates. Of course, the set of tangent spaces (at all points of Q) to the fibers of a fibration form a differential system.

Foliations.

More generally, we say that a differential system $\mathfrak{D}$ is a **foliation** or is **completely integrable** if it is tangent to to a foliation in the preceding sense: Every point $q \in Q$ has a coordinate neighborhood U with coordinates $x_1, \ldots, x_n$ with q corresponding to $x = 0$ and a map $f : U \to \mathbb{R}^{n-k}$ with

$$f(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-k})$$

and such that at every point q of U, the space $\mathfrak{D}_q$ is the tangent space to the fiber of f through q.

All one dimensional distributions are foliations.

For example, every one dimensional differential system is completely integrable. This is an immediate consequence of the existence theorem for ordinary differential equations. But not every one dimensional foliation is a submersion. For example, the irrational line foliation on a two dimensional torus.

As we have seen, the two dimensional differential system on $\mathbb{R}^3$ defined by the null planes of $dz + xdy$ is *not* completely integrable. The Frobenius theorem to be stated below (in two versions) gives a useful necessary and sufficient condition for a differential system to be a foliation.

Fibrations.

A more stringent condition on a submersion is for $f : Q \to B$ to be a **fibration**. We say that f is a fibration if it is surjective and has the local triviality condition: There exists a manifold F (called the standard fiber) such that about every point in B there is a neighborhood W and a diffeomorphism $\phi : f^{-1}(W) \to W \times F$ which conjugates f into projection to the first factor.

The vector fields of a differential system.

Let $\mathfrak{D}$ be a differential system on some manifold M. We let $\mathfrak{X}(\mathfrak{D})$ denote the space of vector fields X on M with the property that $X_p \in \mathfrak{D}_p$ at all p.

Theorem 1 Theorem of Frobenius. *A necessary and sufficient condition for $\mathfrak{D}$ to be completely integrable is that $\mathfrak{X}(\mathfrak{D})$ be closed under Lie bracket. i.e.*

$$X, Y \in \mathfrak{X}(\mathfrak{D}) \;\Rightarrow [X, Y] \in \mathfrak{X}(\mathfrak{D}).$$

Proof of necessity.

Proof of necessity. If $\mathfrak{D}$ is completely integrable of dimension k, every point has a coordinate neighborhood with coordinates $x_1, \ldots, x_n$ such that

$$\frac{\partial}{\partial x_i} \in \mathfrak{X}(\mathfrak{D}) \quad \text{for } i = 1, \ldots k.$$

So the values of these vector fields span $\mathfrak{D}_p$ at each point of the neighborhood and hence any element of $\mathfrak{X}(\mathfrak{D})$ is a sum of elements of $\mathfrak{X}(\mathfrak{D})$ of the form $a\frac{\partial}{\partial x_i}$ for some $1 \leq i \leq k$ and where a is smooth function. But

$$\left[a\frac{\partial}{\partial x_i}, b\frac{\partial}{\partial x_j}\right] = a\frac{\partial b}{\partial x_i}\frac{\partial}{\partial x_j} - b\frac{\partial a}{\partial x_j}\frac{\partial}{\partial x_i}$$

lies in $\mathfrak{X}(\mathfrak{D})$. $\square$

Sufficiency.

Proof of sufficiency. This will be by induction on k. We know the theorem to be true for $k = 1$ where the condition is vacuous and all differential systems are completely integrable.

Let $x \in M$ and let $X_1, \ldots, X_k$ be vector fields defined in a neighborhood of x and whose values at each point p in the neighborhood span $\mathfrak{D}_p$. Let S be a submanifold of dimension $n-1$ passing through x and such that X_1 is nowhere tangent to S. Solve the ordinary differential equation corresponding to X_1 with initial conditions for the trajectories to lie in S at time 0. If $y_2, \ldots y_n$ are coordinates on S, and if $y_1(p)$ denotes the time at which a trajectory starting at S reaches p, then the variables $y_1, \ldots y_n$ are coordinates about x in terms of which

$$X_1 = \frac{\partial}{\partial y_1}.$$

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Let

$$f_i := X_i y_1$$

and set

$$Y_1 = X_1, \qquad Y_i := X_i - f_i X_1, \qquad i = 2, \dots, k.$$

Then the vector fields $Y_1, \dots Y_p$ are still linearly independent at all points in a neighborhood about x and span $\mathfrak{D}$ in this neighborhood, and, in addition,

$$Y_i y_1 = 0, \qquad i = 2, \dots, p.$$

The preceding equation implies that the Y_i are all tangent to the initial hypersurface S given by $y_1 = 0$. So there are vector fields $Z_2, \ldots, Z_k$ on S such that

$$d\iota_q(Z_{iq}) = Y_{iq}$$

at all $q \in S$, where $\iota : S \to M$ denotes the injection of S into M. The vector fields $Z_2, \ldots, Z_k$ define a differential system of dimension $k-1$ on S which satisfies Frobenius's criterion that $[Z_i, Z_j]$ lie in the subspace spanned by the Z_i at all points. Indeed, if this were not true at some point q of S then the the same would hold for $[Y_i, Y_j]$ since none of the Y_i have any component in $\frac{\partial}{\partial y_1}$ direction.

The induction.

So by induction we conclude that we can find coordinates $w_2, \ldots, w_n$ on S near x so that the differential system spanned by $Z_2, \ldots, Z_k$ is tangent to the foliation given by projection onto the last $n-k-1$ coordinates. Then

$$(x_1, \ldots, x_n) = (y_1, w_2, \ldots, w_n)$$

is a system of coordinates about x in M and

$$\frac{\partial x_i}{\partial y_1} = 0$$

so

$$Y_1 = \frac{\partial}{\partial y_1} = \frac{\partial}{\partial x_1}.$$

In particular,

$$Y_1 x_s = 0 \quad \text{for} \quad s = k+1, \ldots, n.$$

Using the condition.

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Therefore there are functions c_{ij} such that

$$\begin{aligned}
\frac{\partial}{\partial x_1}(Y_i x_s) &= Y_1(Y_i x_s) \\
&= Y_1(Y_i x_s) - Y_i(Y_1 x_s) \\
&= [Y_1, Y_i] x_s \\
&= c_{i1} Y_1 x_s + \sum_{j=2}^{k} c_{ij} Y_j x_s \ \ \text{since } [Y_1, Y_i] \in \mathfrak{X}(\mathfrak{D}) \\
&= \sum_{j=2}^{k} c_{ij} Y_j x_s.
\end{aligned}$$

So the $Y_i x_s$ satisfy the system of (ordinary) homogeneous linear differential equations

$$\frac{\partial}{\partial x_1}(Y_i x_s) = \sum_{j=2}^{k} c_{ij} Y_j x_s$$

with the initial conditions at $x_1 = 0$ given by

$$Y_i x_s = Z_i x_s = 0.$$

The uniqueness theorem for differential equations then implies that

$$Y_i x_s \equiv 0, \quad i \leq k, \quad s > k.$$

So the vector fields Y_i are function linear combinations of

$$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}$$

and since the $Y_1, \dots, Y_k$ are linearly independent everywhere, this shows that $\mathfrak{D}$ is completely integrable. $\square$

Our main application of Frobenius.

Let Ω be a closed form of any degree. (We will be interested in forms of degree 2.) Consider the set of vector fields which satisfy

$$i(X)\Omega = 0.$$

This may or may not define a differential system, since the dimension of the space spanned by such X_p may vary with p. Suppose that this dimension is constant. Then I claim that the differential system we get is a foliation. Indeed, if $i(X)\Omega = 0$ then Weil's formula implies that

$$D_X\Omega = di(X)\Omega = i(X)d\Omega = 0.$$

Hence if $i(Y)\Omega = 0$ then

$$i([X, Y])\Omega = D_X\left(i(Y)\Omega\right) = 0.$$

We call this foliation, that is the foliation spanned by the vector fields satisfying $i(X)\Omega = 0$ **null foliation** of Ω.

Cartan's generalization.

There is a very pretty generalization of this fact due Cartan: Let $\mathfrak{I}$ be an ideal in the ring of differential forms on a manifold M which is homogeneous: if $\sigma \in \mathfrak{I}$ then all the homogeneous components of σ belong to $\mathfrak{I}$. Consider the set of vector fields X on M which satisfy

$$i(X)\mathfrak{I} \subset \mathfrak{I}.$$

In other words, the set of all vector fields $X$with the property that

$$\sigma \in \mathfrak{I} \quad \Rightarrow \quad i(X)\sigma \in \mathfrak{I}.$$

Once again, this may or may not define a differential system, since the dimension of the space spanned by the X_p may vary from one p to another.

The characteristic system of an ideal.

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Once again, this may or may not define a differential system, since the dimension of the space spanned by the X_p may vary from one p to another. Suppose that this dimension is constant. So we get a differential system which is called the **characteristic system** of the ideal $\mathfrak{I}$.

Cartan's theorem.

Theorem 2 [Cartan.] *If $d\mathfrak{I} \subset \mathfrak{I}$, its characteristic system is completely integrable.*

The condition $d\mathfrak{I} \subset \mathfrak{I}$ means that if $\sigma \in \mathfrak{I}$ then $d\sigma \in \mathfrak{I}$.

Proof. If $i(X)\mathfrak{I} \subset \mathfrak{I}$ and $\sigma \in \mathfrak{I}$ then

$$D_X\sigma = i(X)d\sigma + di(X)\sigma \in \mathfrak{I}.$$

So if $i(Y)\sigma \in \mathfrak{I}$ then $D_X(i(Y)\sigma)) \in \mathfrak{I}$. But

$$D_X(i(Y)\sigma)) = i([X,Y])\sigma + i(Y)D_X\sigma$$

so if $i(X)\mathfrak{I} \subset \mathfrak{I}$ and $i(Y)\mathfrak{I} \subset \mathfrak{I}$ then $i([X,Y])\mathfrak{I} \subset \mathfrak{I}$. $\square$

Horizontal and basic forms of a foliation.

Let $\pi : Q \to B$ be a foliation. If τ is a differential form on B then $\sigma = \pi^*\tau$ is a differential form on Q with the following two properties:

- It is **horizontal** in the sense that if X is a vector field which is everywhere tangent to the fiber (we say that X is **vertical**) then
$$i(X)\sigma = 0$$
and
- It is vertically invariant in the sense that $D_X\sigma = 0$ for every vertical vector field.

Conversely, suppose that σ satisfies the first condition. Let us introduce local product coordinates $(x_1, \ldots, x_f, y_1, \ldots, y_b)$ where $(x_1, \ldots, x_f)$ are local coordinates on the fiber F and $(y_1, \ldots, y_b)$ are local coordinates on the base B. If σ is horizontal then σ must be a linear combination with function coefficients of products of the dy's. These functions a might depend on all the variables. But if σ also vertically invariant, they must satisfy

$$\frac{\partial a}{\partial x_i} \equiv 0. \quad i = 1, \ldots, f.$$

In other words they are locally constant in the fiber direction. If F is connected, this implies that $\sigma = \pi^* \tau$ for some form τ on B. We say that σ is **basic** in the sense that it come from the base. We have proved:

Proposition 1 *Let $Q \to B$ be a fibration with connected fibers. Then a differential form on Q is basic if and only if it is horizontal and is vertically invariant.*

Reduction of a closed form.

Let Ω be a closed form on a manifold Q and consider the set of vector fields which satisfy

$$i(X)\Omega = 0.$$

Recall that this may or may not define a differential system, since the dimension of the space spanned by such X_p may vary with p. Suppose that this dimension is constant. We say that Ω has **constant rank**. Then we know that the differential system that we get is a foliation $\mathfrak{D}$ called the **null foliaton**. Furthermore, by definition, Ω is horizontal with respect to this foliation in the sense that if $X \in \mathfrak{X}(\mathfrak{D})$ then $i(X)\Omega = 0$ and is vertically invariant with respect to this foliation in the sense that if $X \in \mathfrak{X}(\mathfrak{D})$ then $D_X\omega = 0$. This follows from Weil's formula as we have seen.

Suppose that this foliation is a fibration $\pi : Q \to B$ with connected fibers. Then by the preceding proposition, we know that $\Omega = \pi^*\omega$ for a uniquely determined form ω on B. Since π^* is an injection and $0 = d\Omega = d\pi^*\omega = \pi^* d\omega$ we conclude that $d\omega = 0$. Finally, I claim that ω is non-degenerate in the following sense: For any $b \in B$ and $v \in T_bB$,

$$i(v)\omega_b = 0 \quad \Rightarrow v = 0.$$

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Indeed, let us introduce local product coordinates

$$(x_1, \ldots, x_f, y_1, \ldots, y_b)$$

around a point $q = (f, b)$ as above. If $i(v)\omega_b = 0$, then the vector $w \in T_q Q \cong T_f F \oplus T_b B$ given by

$$w = (0, v)$$

satisfies

$$i(w)\Omega_q = 0.$$

So $w \in \mathfrak{D}_q$. But all vectors of $\mathfrak{D}_q$ are vertical, i.e of the form $(z, 0)$. So $v = 0$.

We have proved:

Proposition 2 *Let Ω be a closed form of constant rank on a manifold Q. Suppose that its null foliation is a fibration $\pi : Q \to B$ with connected fibers. Then*

$$\Omega = \pi^* \omega$$

where ω is a non-degenerate closed form on B.

Reduction of a co-isotropic immersion.

I want to apply the the preceding proposition to the following situation. (M, ϖ) is a symplectic manifold and

$$\iota : Q \to M$$

is a co-isotropic immersion. This means that ι is an immersion such that

$$d\iota_q(T_qQ) \subset T_{\iota(q)}M$$

is a co-isotropic subspace relative to the symplectic form $\varpi_{\iota(q)}$ on $T_{\iota(q)}M$ for every $q \in Q$. Recall that being co-isotropic means that

$$d\iota_q(T_qQ)^\perp \subset d\iota_q(T_qQ).$$

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Now $\dim d\iota_q(T_qQ)^\perp = \dim M - \dim Q$ and so is constant. Since ι is an immersion, the dimension of

$$(d\iota_q)^{-1}\left(d\iota_q(T_qQ)\right) \subset T_qQ$$

is constant, and so we get a differential system on Q. By definition, this is the differential system associated to the closed two form

$$\Omega = \iota^*\varpi,$$

i.e. the null-foliation of Ω. By abuse of language we will refer to this as the null foliation of Q (where ι is understood). We conclude:

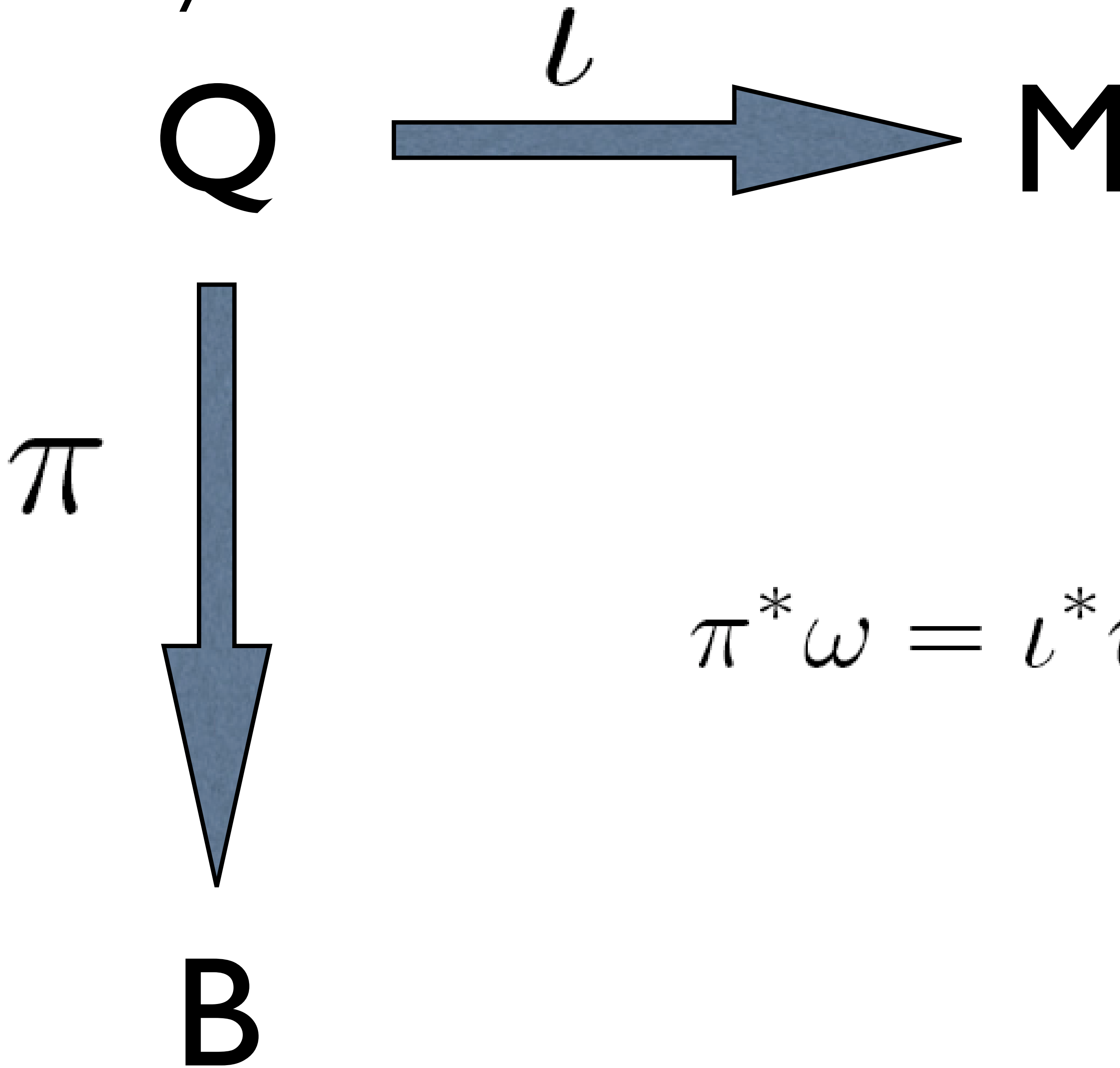
Theorem 3 *Suppose that the null foliation of Q is a fibration*

$$\pi : Q \to B$$

with connected fibers. The B is a symplectic manifold with symplect form ω such that

$$\pi^*\omega = \iota^*\varpi. \tag{1}$$

Diagramatically we have



$$\pi^*\omega = \iota^*\varpi.$$

I now want to study a relation between Poisson brackets on M and Poisson brackets on B. Let f be a smooth function on M so that X_f denotes its corresponding Hamiltonian vector field which satisfies $i(X_f)\varpi = df$. So if $q \in Q$ and $v \in T_q Q$ we have

$$\langle df_{\iota(q)}, d\iota_q(v)\rangle = \varpi_{\iota(q)}\left(X_f(\iota(q)), d\iota_q(v)\right).$$

Suppose we choose $v \in \mathfrak{D}_q$ where $\mathfrak{D}$ is the null foliation of Q. Since the orthocomplement of $d\iota_q(\mathfrak{D}_q)$ relative to ϖ_q is $d\iota_q(T_q Q)$ we see that

Proposition 3 *$d(\iota^* f)_q$ vanishes on $\mathfrak{D}_q$ if and only if $X_f(q) \in d\iota_q(T_q Q)$.*

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Applied to all points of Q we see that

Proposition 4 *The function $\iota^* f$ on Q is constant along the null foliation of Q if and only if X_f is tangent to the image of Q in M.*

If the null foliation is a fibration with connected fibers, then a function is constant alont the null fibration if an only if it is basic, i.e. of the form

$$\pi^* F$$

where F is a function on the base B.

So if f is a function on M such that X_f is tangent to the image of Q, we have

$$\iota^* f = \pi^* F$$

for a (unique smooth) function F on B. Suppose that this is the case. Suppose that $v \in T_q Q$ and let $b = \pi(q)$ and $m = \iota(q)$. Then

$$\langle df(m), d\iota_q(v)\rangle = \langle dF(b), d\pi_q(v)\rangle = \omega_b(Y_F, d\pi_q(v))$$

where Y_F is the Hamiltonian vector field on B corresponding to the function F and the symplectic form ω.

$$\langle df(m), d\iota_q(v)\rangle = \langle dF(b), d\pi_q(v)\rangle = \omega_b(Y_F, d\pi_q(v))$$

where Y_F is the Hamiltonian vector field on B corresponding to the function F and the symplectic form ω. Now

$$\langle df(m), d\iota_q(v)\rangle = \Omega_{\iota(q)}\left(X_f(q), d\iota_q(v)\right) = \omega_b\left(d\pi_q\left(X_f(q)\right), d\pi_q v\right)$$

where we have written $X_f(q)$ instead of $(d\iota_q)^{-1}(X_f(q))$.

Since the vectors of the form $d\pi_q v$ span T_bB we see that

$$d\pi_q(X_f)(q) = Y_F(b).$$

So if f_1 and f_2 are two functions on M such that their Hamiltonian vector fields are tangent to $\iota(Q)$ we get

$$\{f_1.f_2\}_M(\iota(q)) = \{F_1, F_2\}_B(\pi(q)).$$

To summarize:

Theorem 4 *Let $\iota : Q \to M$ be a coisotropic immersion whose null foliation is a fibration $\pi : Q \to B$. If f_1 and f_2 are two functions on M such that $\iota^* f_i = \pi^* F_i$ for $i = 1, 2$ then*

$$\iota^*\{f_1, f_2\}_M = \pi^*\{F_1, F_2\}_B.$$