Symplectic geometry
Lecture 8

1. The Darboux-Weinstein theorems.

These are theorems which state that two symplectic structures on a manifold are the same or give a normal form near a submanifold etc. via the Moser-Weinstein method.

2. Some group theory.

This method hinges on the generalized Weil formula which we now recall:

If $f_t : X \to Y$ is a smooth family of maps and ω_t is a one parameter family of differential forms on Y then

$$\frac{d}{dt} f_t^* \omega_t = f_t^* \frac{d}{dt} \omega_t + Q_t d\omega_t + dQ_t \omega_t \tag{1}$$

where

$$Q_t : \Omega^k(Y) \to \Omega^{k-1}(X)$$

is given by

$$Q_t \sigma(w_1, \ldots, w_{k-1}) := \sigma(v_t, df_t(w_1), \ldots, df_t(w_{k-1}))$$

where

$$v_t : X \to T(Y), \quad v_t(x) := \frac{d}{dt} f_t(x).$$

The homotopy formula.

$$\frac{d}{dt} f_t^* \omega_t = f_t^* \frac{d}{dt} \omega_t + Q_t d\omega_t + dQ_t \omega_t \qquad (1)$$

If ω_t does not depend explicitly on t then the first term on the right of (1) vanishes, and integrating (1) with respect to t from 0 to 1 gives the **homotopy formula**

$$f_1^* - f_0^* = dQ + Qd, \quad Q := \int_0^1 Q_t dt. \qquad (2)$$

Darboux-Weinstein version 1: compact manifolds.

Theorem 1 *Let M be a compact manifold, ω_0 and ω_1 two symplectic forms on M in the same cohomology class so that*

$$\omega_1 - \omega_0 = d\alpha$$

for some one form α. Suppose in addition that

$$\omega_t := (1-t)\omega_0 + t\omega_1$$

is symplectic for all $0 \leq t \leq 1$. Then there exists a diffeomorphism $f : M \to M$ such that

$$f^*\omega_1 = \omega_0.$$

Proof. Solve the equation

$$i(v_t)\omega_t = -\alpha$$

which has a unique solution v_t since ω_t is symplectic. Then solve the time dependent differential equation

$$\frac{df_t}{dt} = v_t(f_t), \quad f_0 = \mathrm{id}$$

which is possible since M is compact. Since

$$\frac{d\omega_t}{dt} = d\alpha,$$

the fundamental formula (1) gives

$$\frac{df_t^*\omega_t}{dt} = f_t^*\left[d\alpha + 0 - d\alpha\right] = 0$$

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so

$$f_t^*\omega_t \equiv \omega_0.$$

In particular, set $t = 1$. QED

Darboux-Weinstein version 2: compact manifolds.

Theorem 2 *Let M be a compact manifold, and ω_t, $0 \le t \le 1$ a family of symplectic forms on M in the same cohomology class.*
Then there exists a diffeomorphism $f : M \to M$ such that

$$f^*\omega_1 = \omega_0.$$

Proof. Break the interval $[0, 1]$ into subintervals by choosing $t_0 = 0 < t_1 < t_2 < \cdots < t_N = 1$ and such that on each subinterval the "chord" $(1 - s)\omega_{t_i} + s\omega_{t_{i+1}}$ is close enough to the curve $\omega_{(1-s)t_i + st_{i+1}}$ so that the forms $(1-s)\omega_{t_i} + s\omega_{t_{i+1}}$ are symplectic. Then successively apply the preceding theorem. QED

Darboux-Weinstein version 3: compact submanifolds.

The next version allows M to be non-compact but has to do with with behavior near a compact submanifold. We will want to use the following proposition:

Proposition 1 *Let X be a compact submanifold of a manifold M and let*

$$i : X \to M$$

denote the inclusion map. Let $\sigma \in \Omega^k(M)$ be a k-form on M which satisfies

$$\begin{aligned} d\sigma &= 0 \\ i^*\sigma &= 0. \end{aligned}$$

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$$\begin{aligned} d\sigma &= 0 \\ i^*\sigma &= 0. \end{aligned}$$

Then there exists a neighborhood U of X and a $k-1$ form β defined on U such that

$$\begin{aligned} d\beta &= \sigma \\ \beta_{|X} &= 0. \end{aligned}$$

(This last equation means that at every point $p \in X$ we have

$$\beta_p(w_1, \ldots, w_{k-1}) = 0$$

for all tangent vectors, not necessarily those tangent to X. So it is a much stronger condition than $i^*\beta = 0$.)

Normal neighborhoods.

For the proof I want to use and generalize the exponential map we defined in the last lecture: Choose a Riemann metric (any Riemann metric) on M. At each $p \in X$, the tangent space T_pM decomposes into an orthogonal direct sum

$$T_pM = T_pX \oplus N_p(X)$$

consisting of the vectors tangent to X and normal to X. Let $N(X)$ be the vector bundle over X consisting of the union of all the $N_p(X)$, $p \in X$. Then we define an exponential map Exp mapping some neighborhood of the zero section of $N(X)$ into M, sending

$$v \in N_p(X) \mapsto \exp v$$

where the exp on the right is the exponential map at p. (Notice that if X is a point, then Exp $=$ exp.)

The map Exp is the identity map when restricted to X (identified as the zero section of $N(X)$) and so the differential of Exp on all vectors in T_pM where $p \in X$ is the identity in view of the above direct sum decomposition. The implicit function theorem then implies that Exp is a diffeomorphism of some neighborhood of the zero section in $N(X)$ onto some neighborhood of X. Now we have a retraction: $v \mapsto tv$ of $N(X)$ onto its zero section and hence of a neighborhood of X in M onto X. Intuitively, choose a sufficiently small neighborhood of X such that if q is in this neighborhood there is a unique closest point $p \in X$ to q. Then retract along the geodesic joining q to p.

Proof of the Proposition. By choice of a Riemann metric and its exponential map, we may find a neighborhood of W of X in M and a smooth retract of W onto X, that is a one parameter family of smooth maps

$$r_t : W \to W$$

and a smooth map $\pi : W \to X$ with

$$r_1 = \mathrm{id}, \quad r_0 = i \circ \pi,\ \pi : W \to X,\ \ r_t \circ i \equiv i.$$

Write

$$\frac{dr_t}{dt} = w_t \circ r_t$$

and notice that $w_t \equiv 0$ at all points of X. Hence the form

$$\beta := Q\sigma$$

has all the desired properties where Q is as in the homotopy formula (2). QED

Theorem 3 *Let X, M and i be as above, and let ω_0 and ω_1 be symplectic forms on M such that*

$$i^*\omega_1 = i^*\omega_0$$

and such that

$$(1-t)\omega_0 + t\omega_1$$

is symplectic for $0 \leq t \leq 1$. Then there exists a neighborhood U of M and a smooth map

$$f : U \to M$$

such that

$$f_{|X} = id \quad \text{and} \quad f^*\omega_0 = \omega_1.$$

Proof. Use the proposition to find a neighborhood W of X and a one form α defined on W and vanishing on X such that

$$\omega_1 - \omega_0 = d\alpha$$

on W. Let v_t be the solution of

$$\iota(v_t)\omega_t = -\alpha$$

where $\omega_t = (1-t)\omega_0 + t\omega_1$. Since v_t vanishes identically on X, we can find a smaller neighborhood of X if necessary on which we can integrate v_t for $0 \leq t \leq 1$ and then apply the Moser argument as above. QED

A variant of the above is to assume that we have a curve of symplectic forms ω_t with $i^*\omega_t$ independent of t.

Darboux-Weinstein version 4: compact submanifolds.

Finally, a very useful variant is Weinstein's

Theorem 4 *X, M, i as above, and ω_0 and ω_1 two symplectic forms on M such that $\omega_{1|X} = \omega_{0|X}$. Then there exists a neighborhood U of M and a smooth map*

$$f : U \to M$$

such that

$$f_{|X} = id \quad \text{and} \quad f^*\omega_0 = \omega_1.$$

Here we can find a neighborhood of X such that

$$(1-t)\omega_0 + t\omega_1$$

is symplectic for $0 \leq t \leq 1$ since X is compact. QED

Darboux's theorem.

One application of the above is to take X to be a point. The theorem then asserts that all symplectic structures of the same dimension are locally symplectomorphic. This is the original theorem of Darboux.

The isotropic embedding theorem.

Another important application of the preceding theorem is Weinstein's isotropic embedding theorem: Let (M, ω) be a symplectic manifold, X a compact manifold, and $i : X \to M$ an isotropic embedding, which means that $di_x(TX)_x$ is an isotropic subspace of $TM_{i(x)}$ for all $x \in X$. Thus

$$di_x(TX)_x \subset (di_x(TX)_x)^\perp$$

where $(di_x(TX)_x)^\perp$ denotes the orthogonal complement of $di_x(TX$ in $TM_{i(x)}$ relative to $\omega_{i(x)}$. Hence

$$(di_x(TX)_x)^\perp / di_x(TX)_x$$

is a symplectic vector space, and these fit together into a symplectic vector bundle (i.e. a vector bundle with a symplectic structure on each fiber).

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$$SN_i(X)$$

or simply by $SN(X)$ when i is taken for granted.

Suppose that U is a neighborhood of $i(X)$ and $g : U \to N$ is a symplectomorphism of U into a second symplectic manifold N. Then $j = g \circ i$ is an isotropic embedding of X into N and f induces an isomorphism

$$g_* : NS_i(X) \to NS_j(X)$$

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of symplectic vector bundles. Weinstein's isotropic embedding theorem asserts conversely, any isomorphism between symplectic normal bundles is in fact induced by a symplectomorphism of a neighborhood of the image:

Theorem 5 *Let* (M, ω_M, X, i) *and* (N, ω_N, X, j) *be the data for isotropic embeddings of a compact manifold* X. *Suppose that*

$$\ell : SN_i(X) \to SN_j(X)$$

is an isomorphism of symplectic vector bundles. Then there is a neighborhood U *of* $i(X)$ *in* M *and a symplectomorphism* g *of* U *onto a neighborhood of* $j(X)$ *in* N *such that*

$$g_* = \ell.$$

For the proof, we will need the following extension lemma:

Proposition 2 *Let*

$$i : X \to M, \quad j : Y \to N$$

be embeddings of compact manifolds X *and* Y *into manifolds* M *and* N. *suppose we are given the following data:*

- *A smooth map* $f : X \to Y$ *and, for each* $x \in X$,
- *A linear map* $A_x TM_{i(x)} \to TN_{j(f(x))}$ *such that the restriction of* A_x *to* $TX_x \subset TM_{i(x)}$ *coincides with* df_x.

Then there exists a neighborhood W *of* X *and a smooth map* $g : W \to N$ *such that*

$$g \circ i = f \circ i$$

and

$$dg_x = A_x \quad \forall\ x \in X.$$

Proof. If we choose a Riemann metric on M, we may identify (via the exponential map) a neighborhood of $i(X)$ in M with a section of the zero section of X in its (ordinary) normal bundle. So we may assume that $M = \mathcal{N}_i X$ is this normal bundle. Also choose a Riemann metric on N, and let

$$\exp : \mathcal{N}_j(Y) \to N$$

be the exponential map of this normal bundle relative to this Riemann metric. For $x \in X$ and $v \in N_i(i(x))$ set

$$g(x, v) := \exp_{j(x)}(A_x v).$$

Then the restriction of g to X coincides with f, so that, in particular, the restriction of dg_x to the tangent space to $T_x X$ agrees with the restriction of A_x to this subspace, and also the restriction of dg_x to the normal space to the zero section at x agrees A_x so g fits the bill. QED

The next stage in the proof of the theorem is to recall and expand upon some results we obtained in symplectic linear algebra about choosing Lagrangian complements:

For any a Lagrangian subspace $L \subset V$ we will need to be able to choose a complementary Lagrangian subspace L', and do so in a consistent manner, depending, perhaps, on some auxiliary data. Here is one such way, depending on the datum of a symmetric positive definite bilinear form B on V. (Here B has nothing to do with with the symplectic form.)

Let L^B be the orthogonal complement of L relative to the form B. So

$$\dim L^B = \dim L = \frac{1}{2} \dim V$$

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and any subspace $W \subset V$ with

$$\dim W = \frac{1}{2}\dim V \quad \text{and } W \cap L = \{0\}$$

can be written as graph (A) where $A : L^B \to L$ is a linear map. That is, under the vector space identification

$$V = L^B \oplus L$$

the elements of W are all of the form

$$w + Aw, \quad w \in L^B.$$

We have

$$\omega(u + Au, w + Aw) = \omega(u, w) + \omega(Au, w) + \omega(u, Aw)$$

since $\omega(Au, Aw) = 0$ as L is Lagrangian. Let C be the bilinear form on L^B given by

$$C(u, w) := \omega(Au, w).$$

Thus W is Lagrangian if and only if

$$C(u, w) - C(w, u) = -\omega(u, w).$$

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Now

$$\mathrm{Hom}(L^B, L) \sim L \otimes L^{B*} \sim L^{B*} \otimes L^{B*}$$

under the identification of L with L^{B*} given by ω. Thus the assignment $A \leftrightarrow C$ is a bijection, and hence the space of all Lagrangian subspaces complementary to L is in one to one correspondence with the space of all bilinear forms C on L^B which satisfy $C(u,w) - C(w,u) = -\omega(u,w)$ for all $u, w \in L^B$. An obvious choice is to take C to be $-\frac{1}{2}\omega$ restricted to L^B. In short,

Proposition 3 *Given a positive definite symmetric form on a symplectic vector space V, there is a consistent way of assigning a Lagrangian complement L' to every Lagrangian subspace L.*

Proposition 3 *Given a positive definite symmetric form on a symplectic vector space V, there is a consistent way of assigning a Lagrangian complement L' to every Lagrangian subspace L.*

Here the word consistent means that the choice depends only on B. This has the following implication: Suppose that T is a linear automorphism of V which preserves both the symplectic form ω and B. In other words, suppose that

$$\omega(Tu, Tv) = \omega(u, v) \quad \text{and} \quad B(Tu, Tv) = B(u, v) \quad \forall\, u, v \in V.$$

Then if $L \mapsto L'$ is the correspondence given by the proposition, then

$$TL \mapsto TL'.$$

More generally, if $T : V \to W$ is a symplectic isomorphism which is an isometry for a choice of positive definite symmetric bilinear forms on each, the above equation holds.

Given L and B (and hence L') we determined the complex structure J by

$$J : L \to L', \quad \omega(u, Jv) = B(u, v) \quad u, v \in L$$

and then

$$J := -J^{-1} : L' \to L$$

and extending by linearity to all of V so that

$$J^2 = -I.$$

Then for $u, v \in L$ we have

$$\omega(u, Jv) = B(u, v) = B(v, u) = \omega(v, Ju)$$

while

$$\omega(u, JJv) = -\omega(u, v) = 0 = \omega(Jv, Ju)$$

so

$$\begin{aligned} J^2 &= -I, \\ \omega(Ju, Jv) &= \omega(u, v), \quad \text{and} \\ \omega(Ju, v) &= \omega(Jv, u). \end{aligned}$$

holds for all $u, v \in V$. We should write $J_{B,L}$ for this complex structure, or J_L when B is understood .

Suppose that T preserves ω and B as above. We claim that

$$J_{TL} \circ T = T \circ J_L$$

so that T is complex linear for the complex structures J_L and J_{TL}. Indeed, for $u, v \in L$ we have

$$\omega(Tu, J_{TL}Tv) = B(Tu, Tv)$$

by the definition of J_{TL}.

To prove: $J_{TL} \circ T = T \circ J_L$

for $u, v \in L$ we have

$$\omega(Tu, J_{TL}Tv) = \omega(Tu, TJ_Lv)$$

showing that

$$TJ_L = J_{TL}T$$

when applied to elements of L. This also holds for elements of L'. Indeed every element of L' is of the form J_Lu where $u \in L$ and $TJ_Lu \in TL'$ so

$$J_{TL}TJ_Lu = -J_{TL}^{-1}TJ_Lu = -Tu = TJ_L(J_Lu).$$

QED

Let I be an isotropic subspace of V and let $I^\perp$ be its symplectic orthogonal subspace so that $I \subset I^\perp$. Let

$$I_B = (I^\perp)^B$$

be the B-orthogonal complement to $I^\perp$. Thus

$$\dim I_B = \dim I$$

and since $I_B \cap I^\perp = \{0\}$, the spaces I_B and I are non-singularly paired under ω. In other words, the restriction of ω to $I_B \oplus I$ is symplectic. The proof of the preceding proposition gives a Lagrangian complement (inside $I_B \oplus I$) to I which, as a subspace of V has zero intersection with $I^\perp$. We have thus proved:

Proposition 4 *Given a positive definite symmetric form on a symplectic vector space V, there is a consistent way of assigning an isotropic complement I' to every co-isotropic subspace $I^\perp$.*

We can use the preceding proposition to prove the following:

Proposition 5 *Let V_1 and V_2 be symplectic vector spaces of the same dimension, with $I_1 \subset V_1$ and $I_2 \subset V_2$ isotropic subspaces, also of the same dimension. Suppose we are given*

- *a linear isomorphism $\lambda : I_1 \to I_2$ and*
- *a symplectic isomorphism $\sigma : I_1^\perp / I_1 \to I_2^\perp / I_2$.*

Then there is a symplectic isomorphism

$$\gamma : V_1 \to V_2$$

such that

1. *$\gamma : I_1^\perp \to I_2^\perp$ and (hence) $\gamma : I_1 \to I_2$,*
2. *The map induced by γ on $I_1^\perp / I_1$ is σ and*
3. *The restriction of γ to I_1 is λ.*

Furthermore, in the presence of positive definite symmetric bilinear forms B_1 on V_1 and B_2 on V_2 the choice of γ can be made in a "canonical" fashion.

Indeed, choose isotropic complements I_{1B} to $I_1^\perp$ and I_{2B} to $I_2^\perp$ as given by the preceding proposition, and also choose B orthogonal complements Y_1 to I_1 inside $I_1^\perp$ and Y_2 to I_2 inside $I_2^\perp$. Then Y_i $(i = 1, 2)$ is a symplectic subspace of V_i which can be identified as a symplectic vector space with $I_i^\perp/I_i$. We thus have

$$V_1 = (I_1 \oplus I_{1B}) \oplus Y_1$$

as a direct sum decomposition into the sum of the two symplectic subspaces $(I_1 \oplus I_{1B})$ and Y_1 with a similar decomposition for V_2. Thus σ gives a symplectic isomorphism of $Y_1 \to Y_2$. Also

$$\lambda \oplus (\lambda^*)^{-1} : I_1 \oplus I_{1B} \to I_2 \oplus I_{2B}$$

is a symplectic isomorphism which restricts to λ on I_1. QED

Proof of Weinstein's istropic embedding theorem. We are given linear maps $\ell_x : (I_x^\perp/I_x) \to J_x^\perp/J_x$ where $I_x = di_x(TX)_x$ is an isotropic subspace of $V_x := TM_{i(x)}$ with a similar notation involving j. We also have the identity map of

$$I_x = TX_x = J_x.$$

So we may apply Proposition 5 to conclude the existence, for each x of a unique symplectic linear map

$$A_x : \ TM_{i(x)} \to TN_{j(x)}$$

for each $x \in X$. We may then extend this to an actual diffeomorphism, call it h on a neighborhood of $i(X)$, and since the linear maps A_x are symplectic, the forms

$$h^*\omega_N \quad \text{and} \quad \omega_M$$

agree at all points of X.

$$h^*\omega_N \quad \text{and} \quad \omega_M$$

agree at all points of X. We then apply Theorem 4 to get a map k such that $k^*(h^*\omega_N) = \omega_M$ and then $g = h \circ k$ does the job. QED

Notice that the constructions were all determined by the choice of a Riemann metric on M and of a Riemann metric on N. So if these metrics are invariant under a group G, the corresponding g will be a G-morphism. If G is compact, such invariant metrics can be constructed by averaging over the group, as will be recalled in the next section.

An important special case of the isotropic embedding theorem.

An important special case of the isotropic embedding theorem is where the embedding is not merely isotropic, but is Lagrangian. Then the symplectic normal bundle is trivial, and the theorem asserts that all Lagrangian embeddings of a compact manifold are locally equivalent, for example equivalent to the embedding of the manifold as the zero section of its cotangent bundle.

Some language from group theory.

If G is a group and X is a set, then an **action** of G on X is a map

$$G \times X \to X, \qquad (a, x) \mapsto ax$$

satisfying $ex = x$ for all $x \in X$ and the "associative law"

$$a(bx) = (ab)x.$$

(This is sometimes called a "left" action but I will try to avoid "right actions".) We also say that "G acts on X".

When more structure is present.

Frequently we demand more: For example, if X is a differentiable manifold, the we demand that the map $x \mapsto ax$ be a smooth map for each $a \in G$. If both G and X are manifolds, then we demand that the map $G \times X \to X$ be smooth, etc. If X is a vector space, we say that the action is **linear** or that we have a **representation** of G on X if the maps $x \mapsto ax$ are linear for each $a \in G$.

G - equivariant maps.

If G acts on X and on Y, then a map

$$F : X \to Y$$

is called G-**equivariant** if

$$F(ax) = aF(x) \quad \forall a \in G,\ x \in X.$$

Another word for a *G* - equivariant map is a *G*-**morphism**.

A special case of equivariance is "invariance". This is when the action of G on Y is trivial ($ay = y$ for all $a \in G,\ y \in Y$). Then the condition reads

$$F(ax) = F(x)\ \forall\ a \in G,\ x \in X.$$

Orbits, isotropy subgroups, topological groups.

If we have an action of G on X and $x \in X$, then the subset of X consisting of all ax, $a \in G$ is known as the G-**orbit** through x. The subgroup of G consisting of all a such that $ax = x$ is known as the **isotropy** group of x and is denoted by G_x.

A group G is a **topological group** if G is a topological space and the multiplication map $G \times G \to G$ is continuous as is the inverse map $a \mapsto a^{-1}$ of $G \to G$.

Averaging over the group.

If G is compact as a topological space then there exists a totally finite invariant measure on G known as Haar measure and denoted by dg. The Haar measure is normalized so that the total measue of G is one. This has the following implication: Suppose that we have a linear action of G on a vector space X. Then for each $x \in X$, we can consider the X-valued function f_x given by $f_x(a) = ax$. If we integrate this function with respect Haar measure, we get an invariant element

$$y = \int_G f_x(g) dg.$$

This procedure is known as **averaging over the group**.

Compact group actions.

Suppose $G \times M \to M$ is a compact group acting smoothly on M. By averaging over the group, we can find a G invariant Riemann

metric on M. Hence, if p is a fixed point, the exponential map gives a G-equivariant diffeomorphism between a neighborhood of the identity in TM_p (under the linearized action of G on this tangent space) and a neighborhood of p in M.

More generally, if X is a compact submanifold invariant under G, the exponential map gives a G-equivariant diffeomorphism between a neighborhood of the zero section in the normal bundle NX of X in M and a G-invariant neighborhood of X in M. Particularly important is the case where X is a G orbit. Then NX is a G-homogeneous vector bundle, that is, it has a G-action preserving the linear structure on each fiber, and G acts transitively on the base. Hence it is isomorphic as a vector bundle to an induced bundle meaning that if $x \in X$, $K := G_x$ and $V = (NX)_x$ with the linear action of K, then

$$NX \sim (G \times V)/K$$

where K acts to the right on G and via its given representation on V. We conclude:

Theorem 6 *If O is a small enough G invariant neighborhood of the origin in V, then $(G \times O)/K$ is equivariantly diffeomorphic to a neighborhood of X in M.*

Mostow's theorem.

The neighborhood O of the origin in V is a "slice" for the G action $(G \times O)/K$ so any point of a neighborhood of X lies on a G-orbit which passes through O. In particular, the isotropy group of such a point is conjugate to the isotropy subgroup of K at a vector in its linear action on V. If v is the vector in question, the isotropy group K_v is the same as the isotropy group K_u where $u = v/\|v\|$ is a point on the unit sphere in V. Hence by induction on the dimension of M we conclude (a special case of) a theorem of Mostow:

Theorem 7 *If a compact group G acts smoothly on a compact manifold M, then, up to conjugacy, only a finite number of subgroups can arise as isotropy subgroups of points of M. In particular, if G is commutative, only a finite number of subgroups of G can arise as isotropy groups of points of M.*

Suppose that K is a compact group acting on a connected manifold M.

Proposition 6 *Suppose that at each $p \in M^K$ the linear isotropy action of K on TM_p is trivial. Then the action of K on M is trivial.*

Proof. The set M^K of fixed points is always closed and K invariant. The hypothesis of the proposition guarantees (via the exponential map) that it contains a neighborhood of each $p \in M^K$. In other words M^K is open. Since M^K is both open and closed, and since M is connected, we conclude that $M^K = M$. QED

Let us now assume that G is compact and commutative, and let K be a minimal element of the list of isotropy groups provided by Mostow's theorem. If $p \in M^K$ the action of K on a neighborhood of p is equivalent to the linear action of K on a neighborhood of of the origin in T_pM. The isotropy group of a point in this neighborhood must thus be a subgroup of K, hence equal to K by its minimality on the list. Hence the linear isotropy action of K is trivial, and thus by the proposition, if M is connected, the action of K on M is trivial. So if action of G on M is faithful (meaning that no element other than the identity acts trivially on M) we conclude that $K = \{e\}$.

So if a compact commutative group acts faithfully on a connected manifold M there exists at least one point $p \in M$ such that $G_p = \{e\}$. Since the isotropy group of all nearby points are a subgroup, we conclude that the set of points whose isotropy group is trivial is an open set. On the other hand, for each of the other isotropy subgroups, the corresponding fixed point set is a union of submanifolds of positive codimension. We have proved:

Theorem 8 *If a compact commutative group acts faithfully on a connected manifold, it acts freely on an open dense subset.*

Symplectic group actions.

Again we begin with the linear theory: Let V be a symplectic vector space. We let $Sp(V)$ denote the group of all all symplectic automorphisms of V, i.e all maps T which satisfy $\omega(Tu, Tv) = \omega(u, v)\ \forall\ u, v \in V$.

A representation $\tau : G \to \mathrm{Aut}(V)$ of a group G is called symplectic if in fact $\tau : G \to Sp(V)$. Our first task will be to show that if G is compact, and τ is symplectic, then we can find a consistent complex structure J which commutes with all the $\tau(a)$, $a \in G$ and such that the associated Hermitian form is

positive definite. Indeed, we can choose a G-invariant element of $\mathrm{Riem}(V)$ and then our polar decomposition argument gives us the desired J.

The space of fixed vectors for a symplectic representation of a compact group is symplectic.

If we choose J as above, if $\tau(a)u = u$ then $\tau(a)Ju = Ju$. So the space of fixed vectors is a complex subspace for the complex structure determined by J. But the restriction of a positive definite Hermitian form to any (complex) subspace is again positive definite, in particular non-singular. Hence its imaginary part, the symplectic form ω, is also non-singular. QED

This result need not be true if the group is not compact. For example, the one parameter group of shear transformations

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

in the plane is symplectic as all of these matrices have determinant one. But the space of fixed vectors is the x-axis.

Symplectic actions.

If $G \times M \to M$ is a symplectic action, and X is a connected component of the fixed point set M^G, then $G_p = G$ for $p \in X$ and T_pX consists of the fixed vectors for the linearized action of G on T_pM. We know that this subspace is symplectic. This means that the restriction of the symplectic form to each tangent space to X is non-degenerate, i.e. that X is a symplectic submanifold.

We have proved

Theorem 9 *Every connected component of the fixed point set M^G of a symplectic action of a compact group is a symplectic submanifold.*