# Symplectic geometry
# Lecture 6

Symplectic manifolds.

Examples.

Hamiltonian mechanics.

# Definition of a symplectic manifold.

**Definition 1** *A* **symplectic manifold** *is a pair* $(M, \omega)$ *consisting of a manifold* $M$ *together with a closed non-degenerate two form* $\omega$ *on* $M$.

Non degeneracy means that for each $m \in M$ the bilinear form $\omega_m$ is a symplectic form on $TM_m$. Equivalently, the top exterior power $\omega^n$ does not vanish anywhere.

The **Liouville form** of $(M, \omega)$ is defined as

$$\Lambda := \frac{1}{n!}\omega^n.$$

# Symplectomorphisms.

**Definition 2** *Given two symplectic manifolds* $(M_1, \omega_1)$ *and* $(M_2, \omega_2)$ *a diffeomorphism*

$$F : M_1 \to M_2$$

*is called a* **symplectomorphism** *if*

$$F^* \omega_2 = \omega_1.$$

The group of all symplectomorphisms of $(M, \omega)$ with itself is denoted by $\mathrm{Symp}(M, \omega)$ or simply by $\mathrm{Symp}(M)$ if $\omega$ is understood.

# Symplectic and Hamiltonian vector fields.

The set of all vector fields $X$ on $M$ such that

$$D_X \omega = 0$$

is denoted by $\mathfrak{X}(M, \omega)$ or by $\mathfrak{X}(M)$ if $\omega$ is understood.

These are called **symplectic** vector fields.

**Definition 3** *For any smooth function $H$ on $M$ the corresponding* **Hamiltonian vector field** *$X_H$ is the unique vector field satisfying*

$$i(X_H)\omega = dH.$$

The space of vector fields $X$ of the form $X = X_H$ for some smooth $H$ is denoted by $\mathfrak{X}_{\mathrm{Ham}}(M, \omega)$ or by $\mathfrak{X}_{\mathrm{Ham}}(M)$ if $\omega$ is understood.

# Hamiltonian vector fields are symplectic.

If $X = X_H$ then Weil's formula gives

$$D_X\omega = di(X)\omega = d(dH) = 0$$

since $d\omega = 0$. So

$$\mathfrak{X}_{\mathrm{Ham}}(M, \omega) \subset \mathfrak{X}(M, \omega).$$

# The bracket of two symplectic vector fields is Hamiltonian.

If $X \in \mathfrak{X}(M,\omega)$ and $Y \in \mathfrak{X}(M,\omega)$ then

$$\begin{aligned} i([X,Y])\omega &= i(D_X Y)\omega \\ &= D_X[i(Y)\omega] - i(Y)D_X\omega \\ &= D_X[i(Y)\omega] \\ &= i(X)di(Y)\omega + di(X)i(Y)\omega \\ &= d[i(X)i(Y)\omega]. \end{aligned}$$

In short the Lie bracket of two elements of $\mathfrak{X}(M,\omega)$ belongs to $\mathfrak{X}_{\mathrm{Ham}}(M,\omega)$:

$$i([X,Y])\omega = d[i(X)i(Y)\omega] = d[-\omega(X,Y)]. \qquad (1)$$

# Poisson brackets.

$$i([X,Y])\omega = d\,[i(X)i(Y)\omega] = d\,[-\omega(X,Y)]\,. \qquad (1)$$

Consider the surjective linear map

$$C^\infty(M) \to \mathfrak{X}_{\mathrm{Ham}}(M) \qquad H \mapsto X_H.$$

The kernel of this map consists of the locally constant functions. In de Rham theory this space is denoted by $H^0(M)$ or by $Z^0(M)$. We use (1) to define the **Poisson bracket** of two $C^\infty$ functions:

$$\{F,G\} := -\omega(X_F, X_G) = i(X_F)i(X_G)\omega = i(X_F)dG = X_F G.$$

This is clearly anti-symmetric in $F$ and $G$ as follows from the first expression: $\{F,G\} := -\omega(X_F, X_G)$. Also, we have the derivation identity:

$$\{F,GH\} = \{F,G\}H + G\{F,H\}$$

which follows from the last expression $\{F,G\} := X_F G$.

$$i([X,Y])\omega = d\left[i(X)i(Y)\omega\right] = d\left[-\omega(X,Y)\right]. \qquad (1)$$

Finally, it follows from (1) in the form

$$[X_F, X_G] = X_{\{F,G\}} \qquad (2)$$

and Weil's formula that the Jacobi identity is satisfied:

$$\{F,\{G,H\}\} = \{\{F,G\},H\} + \{G,\{F,H\}\}.$$

Indeed,

$$\begin{aligned} \{F,\{G,H\}\} &= X_F\{G,H\} \\ &= D_{X_F}\{G,H\} \\ &= -D_{X_F}(\omega(X_G,X_H)) \end{aligned}$$

$$
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\end{aligned}
$$

$$
= -(D_{X_F}\omega)(X_G,X_H) - \omega(D_{X_F}X_G,X_H) - \omega(X_G,D_{X_F}X_H)
$$

by Leibnitz's rule

$$
\begin{aligned}
&= -\omega([X_F,X_G],X_H) - \omega(X_G,[X_F,X_H]) \quad \text{since} \quad D_{X_F}\omega = 0 \\
&= -\omega(X_{\{F,G\}},X_H) - \omega(X_G,X_{\{F,H\}}) \quad \text{by (2)} \\
&= \{\{F,G\},H\} + \{G,\{F,H\}\}. \quad \Box
\end{aligned}
$$

So the Poisson bracket gives a Lie algebra structure on $C^\infty(M)$ and the map $F \mapsto X_F$ is a surjective Lie algebra homomorphism

$$
C^\infty(M) \to \mathfrak{X}_{\mathrm{Ham}}(M).
$$

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$$C^\infty(M) \to \mathfrak{X}_{\mathrm{Ham}}(M).$$

The locally constant functions are in the center of $C^\infty(M)$. In particular they form a trivial Lie algebra and we have the exact sequence

$$0 \to Z^0(M) \to C^\infty(M) \to \mathfrak{X}_{\mathrm{Ham}}(M) \to 0$$

of Lie algebras.

# Poisson algebras.

A **Poisson algebra** is defined to be a commutative algebra (under a multiplication denoted by $m(F, G) = FG$) which is also a Lie algebra where the bracket of $F$ with $G$ is denoted by $\{F, G\}$, and such that the Lie bracket acts as derivations of the commutative multiplication. So we can say that the smooth functions on a symplectic manifold form a Poisson algebra.

# Siméon Denis Poisson

**Born:** 21 June 1781 in Pithiviers, France
**Died:** 25 April 1840 in Sceaux (near Paris), France

# Poisson manifolds.

More generally, we say that a manifold is a Poisson manifold if its space of smooth functions is equipped with the structure of Poisson algebra. Every symplectic manifold is a Poisson manifold, but the converse is not true. For example, we could define a Poisson structure on any manifold by setting the Poisson bracket of any two functions to be zero. As a less trivial example, we can consider the case where $N = M \times S$ where $M$ is a symplectic manifold and where $S$ is any auxiliary manifold. Any two smooth functions on $N$ restrict, for each $s \in S$ to smooth functions on $M$ where we can take their Poisson bracket which then gives a function on $N$.

# The basic local example.

Consider a symplectic vector space $V$ of dimension $2n$. Let $q_1, \ldots q_n, p_1, \ldots p_n$ be (linear) coordinates with respect to a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$. In any vector space, we have the identification of the tangent space at every point $x$ with $V$. This means that any vector in $V$ can be identified with a vector field. In our case, the vectors $e_i$ are identified with the vector fields $\partial/\partial q_i$ and the vectors $f_j$ are identified with $\partial/\partial p_j$. This implies that in the $q, p$ coordinates we have

$$\omega = \sum_i dq_i \wedge dp_i$$

and hence the Liouville form is

$$\Lambda = dq_1 \wedge dp_1 \wedge dq_2 \wedge dp_2 \wedge \cdots \wedge dq_n \wedge dp_n.$$

By restriction, we can consider the form $\omega = \sum_i dq_i \wedge dp_i$ on any open set $U$ of $V$. (We will see later (Darboux's theorem) that every symplectic form is locally symplectomorphic to this example.)

Given a smooth function $H$ on $U$, let

$$X_H = \sum_i \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right). \tag{3}$$

Then

$$i(X_H)\omega = \sum_i \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i = dH.$$

In other words, $X_H$ is the Hamiltonian vector field associated to $H$ as required by the notation.

# Hamilton's equations.

$$X_H = \sum_i \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right).$$

Hence the ordinary differential equations defined by $H$ are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

These are the famous Hamilton equations of classical mechanics.

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

If $U = V$ and $H = p_j$ these equations become

$$\dot{p}_j = 0,\ \dot{q}_i = 0, i \neq j,\ \dot{q}_j = 1.$$

So the flow generated by $X_H$ consists of translation by $t$ in the $e_j$ direction.

Similarly, the flow corresponding to $q_j$ consists of translation by $t$ in the $-f_j$ direction.

# Quadratic Hamiltonians.

Let us now consider the cases where $H$ is a homogeneous quadratic polynomial: We want to consider three types of homogeneous quadratic polynomials:

- Polynomials of "mixed type", that is, linear combinations of $q_i p_j$.
- Polynomials in $p$, that is linear combinations of $p_i p_j$, and
- Polynomials in $q$, that is linear combinations of $q_i q_j$.

# Quadratic Hamiltonians - mixed type.

First consider the polynomial $q_i p_j$. The corresponding $X_H$ given by (3) is

$$q_i \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial q_i}.$$

If we let $E_{ij}$ denote the $n \times n$ matrix with a 1 in the $ij$ position and zeros elsewhere, then this vector field corresponds to the action of the matrix (in $sp(V)$) given by

$$\begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ij}^t \end{pmatrix}.$$

By linearity, we see that the most general polynomial of the form $\sum_{ij} a_{ij} q_i p_j$ corresponds to a matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}.$$

# Polynomials in $p$ .

Similarly, the polynomial $p_i p_j, i \neq j$ corresponds to the matrix

$$\begin{pmatrix} 0 & S_{ij} \\ 0 & 0 \end{pmatrix}$$

where $S_{ij}$ has ones in the $ij$ and $ji$ positions and zeros elsewhere, while $\frac{1}{2}p_i^2$ corresponds to the matrix

$$\begin{pmatrix} 0 & E_{ii} \\ 0 & 0 \end{pmatrix}.$$

So the most general polynomial quadratic in the $p$'s alone corresponds to an element of $sp(V)$ of the form

$$\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$$

were $S$is a symmetric $n \times n$ matrix.

# Polynomials in $q$ .

The quadratic polynomial $q_i q_j$ corresponds to the matrix

$$\begin{pmatrix} 0 & 0 \\ -S_{ij} & 0 \end{pmatrix}$$

and, in general, a quadratic polynomial of the form $\sum a_{ij} q_i q_j$ corresponds to an element of $sp(V)$ of the form

Text

$$\begin{pmatrix} 0 & 0 \\ -S & 0 \end{pmatrix}.$$

By what we observed earlier, the Poisson bracket of quadratic polynomials corresponds to the negative of the bracket on the Lie algebra $sp(V)$.

# The cotangent bundle.

Let $Q$ be an arbitrary smooth manifold. Its cotangent bundle $T^*Q$ is defined (as a set) as the union of all cotangent spaces at all points of $Q$, where the cotangent space $T_x^*Q$ is defined as the space of all linear functions on $T_xQ$. It is routine to check that $T^*Q$ has a natural structure as a manifold, and that the projection $\pi : T^*Q \to Q$ sending every element of $T_x^*Q$ to $x$ is smooth.

## The canonical one form on a cotangent bundle.

If $Q$ is a differentiable manifold, then its cotangent bundle $T^*Q$ carries a **canonical one form** $\alpha = \alpha_Q$ defined as follows: Let

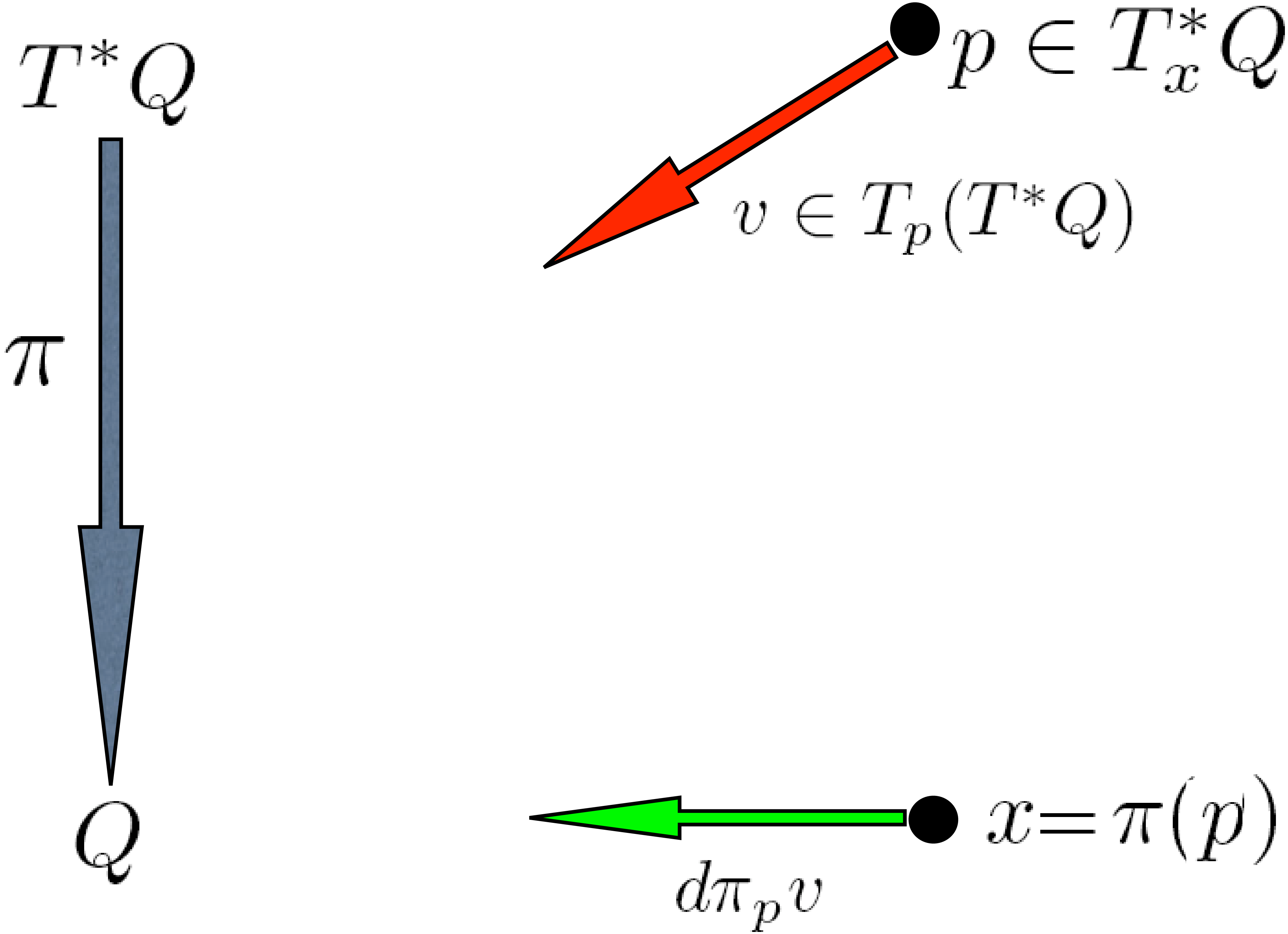
$$\pi : T^*Q \to Q$$

be the projection sending any covector $p \in T_x^*Q$ to its base point $x$. If $v \in T_p(T^*Q)$ is a tangent vector to $T^*Q$ at $p$, then

$$d\pi_p v$$

is a tangent vector to $Q$ at $x$. In other words, $d\pi_p v \in T_x Q$. But $p \in T_x^*Q$ is a linear function on $T_xQ$, and so we can evaluate $p$ on $d\pi_p v$. The canonical linear differential form $\alpha$ is defined by

$$\langle \alpha_p, v \rangle := \langle p, d\pi_p v \rangle \quad \text{if} \quad v \in T_p(T^*Q). \tag{4}$$




$$\langle \alpha_p, v \rangle := \langle p, d\pi_p v \rangle$$

## The canonical two form on the cotangent bundle.

This is defined as

$$\omega_Q = -d\alpha_Q. \tag{5}$$

Let $q^1, \ldots, q^n$ be local coordinates on $Q$. Then $dq^1, \ldots, dq^n$ are differential forms which give a basis of $T_x^*Q$ at each $x$ in the coordinate neighborhood $U$. In other words, the most general element of $T_x^*Q$ can be written as $p_1(dq^1)_x + \cdots + p_n(dq^n)_x$. Thus $q^1, \ldots, q^n, p_1, \ldots, p_n$ are local coordinates on

$$\pi^{-1}U \subset T^*Q.$$

In terms of these coordinates the canonical one-form is given by

$$\alpha = p \cdot dq = p_1 dq^1 + \cdots + p_n dq^n$$

Hence the canonical two-form has the standard local expression

$$\omega = dq \wedge \cdot dp = dq^1 \wedge dp_1 + \cdots + dq^n \wedge dp_n. \tag{6}$$

## Hamilton's equations on the cotangent bundle.

If $H$ is a $C^\infty$ function on $T^*Q$ then we get the corresponding vector field $X_H$ which in terms of the local expression (6) takes on the form of Hamilton's differential equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$

Suppose that $Q = \mathbb{R}^n$ so that these coordinates are in fact global, corresponding to (the standard, say) linear coordinates on $Q$. If

$$H(q,p) = \frac{1}{2m}\sum_i p_i^2$$

where $m > 0$ is the "mass", these equations become

$$\dot{q}_i = \frac{1}{m}p_i, \quad \dot{p}_i = 0.$$

# Galileo.

$$\dot{q}_i = \frac{1}{m} p_i, \quad \dot{p}_i = 0.$$

So if we write $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ so that $\dot{\mathbf{q}}$ is the velocity then the solution curves of this system are

$$\mathbf{q}(t) = \mathbf{q}(0) + t\dot{\mathbf{q}}(0), \quad \mathbf{p}(t) \equiv m\dot{\mathbf{q}}(0).$$

The particle moves along a straight line with constant velocity (Galileo's law) and $\mathbf{p} = m\mathbf{q}$ is the momentum.

# F=ma

If

$$H(q,p) = \frac{1}{2m}\sum_i p_i^2 + V(q)$$

then the above equations are modified so as to become

$$\dot{q}_i = \frac{1}{m}p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}.$$

So if we still interpret $\mathbf{p}$ as momentum, then the differential form on $Q$

$$-dV = \sum_i \frac{\partial V}{\partial q_i} dq_i$$

represents a "force field" and the preceding equations give Newton' law $F = m\mathbf{a}$. Notice that Hamilton's equations give Newton's laws for a "conservative system" where the force field is the differential of a function.

## Lagrangian submanifolds of the cotangent bundle.

A submanifold $\Lambda$ of a symplectic manifold is called **Lagrangian** if its tangent space $T_x\Lambda$ at every point $x \in \Lambda$ is a Lagrangian subspace of $T_xM$. Put another way, the dimension of $\Lambda$ is one half the dimension of $M$ and the restriction on the symplectic form $\omega$ to $\Lambda$ is identically zero.

To say that a submanifold $\Lambda \subset T^*Q$ is Lagrangian means that $\Lambda$ has the same dimension as $Q$ and that the restriction to $\Lambda$ of the canonical one form $\alpha_Q$ is closed.

# Horizontal Lagrangian submanifolds of the cotangent bundle.

Suppose that $Z$ is a submanifold of $T^*Q$ and that the restriction of $\pi : T^*Q \to Q$ to $Z$ is a diffeomorphism. This means that $Z$ is the image of a section

$$s: \; Q \to T^*Q.$$

Giving such a section is the same as assigning a covector at each point of $Q$, in other words it is a linear differential form. For the purposes of the discussion we temporarily introduce a redundant notation and call the section $s$ by the name $\beta_s$ when we want to think of it as a linear differential form. We claim that

$$s^*\alpha_Q = \beta_s.$$

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Indeed, if $w \in T_xQ$ then $d\pi_{s(x)} \circ ds_x(w) = w$ and hence

$$s^*\alpha_Q(w) = \langle (\alpha_Q)_{s(x)}, ds_x(w) \rangle =$$

$$= \langle s(x), d\pi_{s(x)} ds_x(w) \rangle = \langle s(x), w \rangle = \beta_s(x)(w).$$

Thus the submanifold $Z$ is Lagrangian if and only if $d\beta_s = 0$. Let us suppose that $Q$ is connected and simply connected. Then $d\beta = 0$ implies that $\beta = d\phi$ where $\phi$ is determined up to an additive constant.

With some slight abuse of language, let us call a Lagrangian submanifold $\Lambda$ of $T^*Q$ **horizontal** if the restriction of $\pi : T^*Q \to Q$ to $\Lambda$ is a diffeomorphism. We have proved

**Proposition 1** *Suppose that $Q$ is connected and simply connected. Then every horizontal Lagrangian submanifold of $T^*Q$ is given by a section $\gamma_\phi : Q \to T^*Q$ where $\gamma_\phi$ is of the form*

$$\gamma_\phi(x) = d\phi(x)$$

*where $\phi$ is a smooth function determined up to an additive constant.*

## Diffeomorphisms of $Q$ yield symplecomorphisms of $T^*Q$.

Let $Q_1$ and $Q_2$ be manifolds and let $f : Q_1 \to Q_2$ be a diffeomorphism. Then the tangent map

$$df : TQ_1 \to TQ_2$$

is a diffeomorphism and so dually there is a diffeomporphsim

$$F = (df^{-1})^* : T^*Q_1 \to T^*Q_2.$$

In more detail: If $\xi \in T^*_x Q_1$ then $F(\xi) \in T^*_{f(x)} Q_2$ is given by

$$F(\xi) = (d(f^{-1})_{f(x)})^* \xi.$$

$$F(\xi) = (d(f^{-1})_{f(x)})^*\xi.$$

The map $F$ covers the map $f$ in the sense that $\pi_2 \circ F = f \circ \pi_1$ where $\pi_1 : T^*Q_1 \to Q_1$ and $\pi_2 : T^*Q_2 \to Q_2$ are the standard projections. We claim that

$$F^*\alpha_2 = \alpha_1 \tag{7}$$

where $\alpha_1$ is the canonical one form on $T^*Q_1$ and $\alpha_2$ is the canonical one form on $T^*Q_2$. Indeed, this is a matter of chasing through the definitions: Let

$$v \in T_\xi(T^*Q_1), \quad \xi \in T_x^*Q_1.$$

Then

$$\begin{aligned}
\langle F^*\alpha_2, v\rangle &= \langle \alpha_2, dF_\xi(v)\rangle \\
&:= \langle F(\xi), d\pi_{2,F(\xi)} dF_\xi(v)\rangle \\
&= \langle F(\xi), df_x(d\pi_{1,\xi}(v))\rangle \\
&:= \langle (df^{-1})^*_{f(x)}\xi, df_x d\pi_{1,\xi}(v)\rangle \\
&= \langle \xi, d\pi_{1,\xi}(v)\rangle \\
&= \langle \alpha_1, v\rangle. \quad \square
\end{aligned}$$

It follows from (8) that $F$ is a symplectomorphism for the canonical symplectic structures on the cotangent bundles.

It follows from the definition that composition of diffeomorphisms corresponds to composition of the associated symplectomorphisms. In particular, we get a group homomorphism

$$\mathrm{Diff}(Q) \to \mathrm{Symp}(T^*Q)$$

for any manifold $Q$. In particular, every flow on $Q$ gives rise to a flow on $T^*Q$ preserving the canonical one form and hence to a symplectic vector field. We claim that this vector field is Hamiltonian. Let us prove this in slightly more generality:

# Exact symplectic manifolds.

We say that a symplectic manifold $(M,\omega)$ is **exact** if there is a one form $\alpha$ on $M$ such that $\omega = -d\alpha$.

Suppose that $X$ is a vector field on $M$ such that $D_X\alpha = 0$. Then by Weil's formula

$$0 = i(X)d\alpha + di(X)\alpha$$

so

$$i(X)\omega = d\langle\alpha, X\rangle.$$

We see that

$$X = X_H, \quad H = \langle\alpha, X\rangle.$$

We see that

$$X = X_H, \quad H = \langle \alpha, X \rangle.$$

In the case that $M = T^*Q$ with $\alpha$ the canonical one form and $X$ the vector field on $T^*Q$ coming from a vector field $Y$ on $Q$ we see that $X = X_H$ where

$$H(\xi) = \langle \xi, Y(\pi\xi) \rangle.$$

In a local coordinate system with

$$Y = \sum_i Y^i \frac{\partial}{\partial q_i}$$

this reads

$$H(q,p) = \sum p_i Y^i(q).$$

Suppose that $\mathcal{E}$ is a vector field on an exact symplectic manifold with

$$i(\mathcal{E})\omega = -\alpha$$

where $d\alpha = -\omega$. Then

$$D_{\mathcal{E}}\omega = -d\alpha = \omega.$$

For example, consider the one parameter group of transformations on $T^*Q$ consisting of multiplication of the cotangent vectors by $e^t$. In local coordinates, the generator is the "Euler vector field"

$$\mathcal{E} = \sum_i p_i \frac{\partial}{\partial p_i}.$$

Clearly

$$i(\mathcal{E}) \sum_i dq_i \wedge dp_i = -\sum p_i dq_i = -\alpha.$$

As another example, consider a symplectic vector space so that

$$\omega = \sum_i dq_i \wedge dp_i$$

and let

$$\mathcal{E} = \frac{1}{2} \sum_i \left( p_i \frac{\partial}{\partial p_i} + q_i \frac{\partial}{\partial q_i} \right).$$

Then

$$i(\mathcal{E})\omega = \frac{1}{2} \sum_i \left( q_i dp_i - p_i dq_i \right).$$

If we set

$$\alpha := \frac{1}{2} \sum_i \left( p_i dq_i - q_i dp_i \right)$$

then $d\alpha = -\omega$.

Conversely, suppose we have a vector field $\mathcal{E}$ on a symplectic manifold $(M, \omega)$ with $D_{\mathcal{E}}\omega = \omega$. By Weil's formula this says that

$$\omega = di(\mathcal{E})\omega$$

so if we set

$$\alpha := -i(\mathcal{E})\omega$$

then

$$d\alpha = -\omega$$

and $\mathcal{E}$ is the corresponding "Euler vector field".

# Adding a "magnetic field".

Let $\sigma$ be a closed two form on $Q$. We can use $\pi$ to pull this two form back to $T^*Q$ and since $d\pi^* = \pi^* d$ the two form $\pi^*\sigma$ is closed, as is

$$\omega_Q + \pi^*\sigma.$$

I claim that $\omega_Q + \pi^*\sigma$ is symplectic. So I must prove that it is of maximal rank which is the same as saying that $(\omega_Q + \pi^*\sigma)^n$ is nowhere zero. Since $\omega_Q$ and $\pi^*\sigma$ are both even, we can apply the binomial formula

$$(\omega_Q + \pi^*\sigma)^n = \sum_k \binom{n}{k} \omega_Q^{n-k} (\pi^*\sigma)^k$$

$$(\omega_Q + \pi^*\sigma)^n = \sum_k \binom{n}{k} \omega_Q^{n-k}(\pi^*\sigma)^k$$

and it is enough to prove that all terms except the first, that is all terms with $k > 0$, vanish, for then $(\omega_Q + \pi^*\sigma)^n = \omega_Q^n$ which we know is nowhere zero. It is easiest to prove this in local coordinates where

$$\omega_q = \sum_i dq_i \wedge dp_i \quad \text{and} \quad \pi^*\sigma = \sum_{i<j} B_{ij}(q) dq_i \wedge dq_j.$$

But $(\pi^*\sigma)^k$ is then a sum of terms each of which involves the wedge product of $2k$ $dq$'s and $\omega_Q^{n-k}$ is a sum of terms each of which involves the wedge product of $n-k$ $dq$'s. So if $k > 0$ the product $\omega_Q^{n-k} \wedge (\pi^*\sigma)^k$ is a sum of terms involving the wedge product of $n+k$ $dq$'s which must vanish.

# The effect of the "magnetic field".

So $\omega_Q + \pi^*\sigma$ does indeed define a new symplectic structure on $T^*Q$. Let us see what effect this modification of the symplectic structure has in the simple case where $Q = \mathbb{R}^n$ and $H = \frac{1}{2m}\sum_i p_i^2$ in the standard coordinates. We are looking for the vector field

$$X_H = \sum_i a_i \frac{\partial}{\partial q_i} + \sum b_i \frac{\partial}{\partial p_i}$$

such that

$$i(X_H)\left(\sum_i dq_i \wedge dp_i + \sum_{i<j} b_{ij} dq_i \wedge dq_j\right) = \frac{1}{m}\sum p_i dp_i.$$

$$i(X_H)\left(\sum_i dq_i \wedge dp_i + \sum_{i<j} b_{ij} dq_i \wedge dq_j\right) = m \sum p_i dp_i.$$

The only way to pick up $\frac{1}{m}p_i$ as the coefficient of $dp_i$ is to have $a_i = \frac{1}{m}p_i$ as before. So the first half of Hamilton's equations

$$\dot{q}_i = \frac{1}{m}p_i$$

remains unchanged. But now

$$i\left(\frac{1}{m}\sum p_i \frac{\partial}{\partial q_i}\right)(\omega_Q + \pi^*\sigma)$$

$$= \frac{1}{m}\sum_i p_i dp_i + i\left(\frac{1}{m}\sum p_i \frac{\partial}{\partial q_i}\right)\sum B_{ij} dq_i \wedge dq_j$$

$$i\left(\frac{1}{m}\sum p_i\frac{\partial}{\partial q_i}\right)(\omega_Q+\pi^*\sigma)$$

$$=\frac{1}{m}\sum_i p_i dp_i + i\left(\frac{1}{m}\sum p_i\frac{\partial}{\partial q_i}\right)\sum B_{ij}dq_i\wedge dq_j$$

and this second term must be compensated for by the coefficients of the $\frac{\partial}{\partial p_j}$. So we must choose the $b_i$ so that

$$\sum b_i dq_i = i\left(\frac{1}{m}\sum p_i\frac{\partial}{\partial q_i}\right)(\sum B_{ij}dq_i\wedge dq_j).$$

This then modifies the second half of Hamilton's equations to become

$$\dot{p}_i = b_i.$$

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We have introduced a "magnetic force" which now involves the velocity. To see what this looks like in terms of notation used in the standard physics courses let us take $n = 3$, and (for convenience) $m = 1$ and write $\pi^*\sigma$ as

$$\pi^*\sigma = B_3 dq_1 \wedge dq_2 - B_2 dq_1 \wedge dq_3 + B_1 dq_1 \wedge dq_2.$$

Then

$$i\left(p_1\frac{\partial}{\partial q_1} + p_2\frac{\partial}{\partial q_2} + p_3\frac{\partial}{\partial q_3}\right)\pi^*\sigma$$

$$= (B_2p_3 - B_3p_2)dq_1 + (B_3p_{-}B_1p_3)dq_2 + (B_1p_2 - B_2p_1)dq_3.$$

So the force is expressed as the "cross product" of the velocity and the magnetic field.