Symplectic geometry
Lecture 4

The space of Lagrangian subspaces of a fixed symplectic vector space.

Lagrangian complements.

Let V be a symplectic vector space.

Proposition 1 *Given any finite collection of Lagrangian subspaces* $M_1, \ldots, M_k$ *of* V *one can find a Lagrangian subspace* L *such that*

$$L \cap M_j = \{0\}, \quad i = 1, \ldots k.$$

Proof. We can always find an isotropic subspace L with $L \cap M_j = \{0\}, \quad i = 1, \dots k$, for example a line which does not belong to any of these subspaces. Suppose that L is an isotropic subspace with $L \cap M_j = \{0\}$, $\forall j$ and is not properly contained in a larger isotropic subspace with this property. We claim that L is Lagrangian. Indeed, if not, $L^\perp$ is a coisotropic subspace which strictly contains L. Let $\pi : L^\perp \to L^\perp/L$ be the quotient map. Each of the spaces $\pi(L^\perp \cap M_j)$ is an isotropic subspace of the symplectic vector space $L^\perp/L$ and so each of these spaces has positive codimension. So we can choose a line ℓ in $L^\perp/L$ which does not intersect any of the $\pi(L^\perp \cap M_j)$. Then $L' := \pi^{-1}(\ell)$ is an isotropic subspace of $L^\perp \subset V$ with $L \cap M_j = \{0\}$, $\forall\ j$ and strictly containing L, a contradiction. $\square$

In words, given a finite collection of Lagrangian subspaces, we can find a Lagrangian subspace which is transversal to all of them.

Parametrizing Lagrangian complements.

Let M be a Lagrangian subspace, and let $\mathcal{L}(V, M)$ denote the set of Lagrangian subspaces transversal to M. Let $L \in \mathcal{L}(V, M)$ be one such subspace. So the non-degenerate pairing between L and M identifies M with the dual space L^* of L. The vector space decomposition

$$V = L \oplus M$$

means that any subspace $N \subset V$ transversal to M projects injectively to L. In particular, if $\dim N = \frac{1}{2} \dim V$ this means that N is the graph of a linear map $T_N : L \to M = L^*$:

$$N = \{(v, T_N v), \quad v \in L\}.$$

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If $w = (v, T_N v)$ and $w' = (v', T_N v')$ are two elements of N, then

$$\omega(w, w') = \omega(v, T_N v') - \omega(v', T_N v)$$

since L and M are Lagrangian. Now a linear map from L to L^* is the same thing as a bilinear form on L. So we see that N is Lagrangian if and only if this bilinear form is symmetric. We have proved

Proposition 2 *If $M \in \mathcal{L}(V)$ and we choose $L \in \mathcal{L}(V, M)$ then we get an identification of $\mathcal{L}(V, M)$ with $S^2(L)$, the space of symmetric bilinear forms on L.*

So every pair of transverse Lagrangian subspaces gives a coordinate chart on $\mathcal{L}(V)$ which is identified with $S^2(L)$. This gives another proof of the fact that $\mathcal{L}(V)$ is a manifold of dimension $\frac{n(n+1)}{2}$ where $n = \frac{1}{2}\dim V$.

Description in terms of a basis.

Suppose that we choose a basis $e_1, \ldots, e_n$ of L and so we get the dual basis $f_1, \ldots, f_n$ of M. If $N \in \mathcal{L}(V, M)$ then we get a basis $g_1, \ldots g_n$ of N given by

$$g_i = e_i + \sum_j S_{ij} f_j$$

where $S_{ij} = S(e_i, e_j)$ and S is the symmetric bilinear form on L corresponding to N by the Proposition. For later use I record the following fact: Suppose that $N \in \mathcal{L}(V, M)$ and $N' \in \mathcal{L}(V, M)$. Then ω induces a (possibly singular) bilinear form on $N \times N'$. In terms of the bases given above for N and N' we have

$$\omega(g_i, g'_j) = S'_{ij} - S_{ij}.$$

Choosing Lagrangian complements "consistently".

The results of this section are purely within the framework of symplectic linear algebra. Hence their logical place is here. However their main interest is that they serve as lemmas for more geometrical theorems, for example the Weinstein isotropic embedding theorem. The results here all have to do with making choices in a "consistent" way, so as to guarantee, for example, that the choices can be made to be invariant under the action of a group.

For any a Lagrangian subspace $L \subset V$ we will need to be able to choose a complementary Lagrangian subspace L', and do so in a consistent manner, depending, perhaps, on some auxiliary data. Here is one such way, depending on the datum of a symmetric positive definite bilinear form B on V. (Here B has nothing to do with with the symplectic form.)

Let L^B be the orthogonal complement of L relative to the form B. So

$$\dim L^B = \dim L = \frac{1}{2}\dim V$$

and any subspace $W \subset V$ with

$$\dim W = \frac{1}{2}\dim V \quad \text{and } W \cap L = \{0\}$$

can be written as

$$\mathrm{graph}(A)$$

where $A : L^B \to L$ is a linear map. That is, under the vector space identification

$$V = L^B \oplus L$$

the elements of W are all of the form

$$w + Aw, \quad w \in L^B.$$

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We have

$$\omega(u + Au, w + Aw) = \omega(u, w) + \omega(Au, w) + \omega(u, Aw)$$

since $\omega(Au, Aw) = 0$ as L is Lagrangian. Let C be the bilinear form on L^B given by

$$C(u, w) := \omega(Au, w).$$

Thus W is Lagrangian if and only if

$$C(u, w) - C(w, u) = -\omega(u, w).$$

Now

$$\mathrm{Hom}(L^B, L) \sim L \otimes L^{B*} \sim L^{B*} \otimes L^{B*}$$

under the identification of L with L^{B*} given by ω. Thus the assignment $A \leftrightarrow C$ is a bijection, and hence the space of all Lagrangian subspaces complementary to L is in one to one correspondence with the space of all bilinear forms C on L^B which satisfy $C(u,w) - C(w,u) = -\omega(u,w)$ for all $u, w \in L^B$. An obvious choice is to take C to be $-\frac{1}{2}\omega$ restricted to L^B. In short,

Proposition 1 *Given a positive definite symmetric form on a symplectic vector space* V*, there is a consistent way of assigning a Lagrangian complement* L' *to every Lagrangian subspace* L.

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Here the word "consistent" means that the choice depends only on B. This has the following implication: Suppose that T is a linear automorphism of V which preserves both the symplectic form ω and the positive definite symmetric form B. In other words, suppose that

$$\omega(Tu, Tv) = \omega(u, v) \quad \text{and} \quad B(Tu, Tv) = B(u, v) \quad \forall\ u, v \in V.$$

Then if $L \mapsto L'$ is the correspondence given by the proposition, then

$$TL \mapsto TL'.$$

More generally, if $T : V \to W$ is a symplectic isomorphism which is an isometry for a choice of positive definite symmetric bilinear forms on each, the above equation holds.

Given L and B (and hence L') we determined the complex structure J by

$$J : L \to L', \quad \omega(u, Jv) = B(u, v) \quad u, v \in L$$

and then

$$J := -J^{-1} : L' \to L$$

and extending by linearity to all of V so that

$$J^2 = -I.$$

Then for $u, v \in L$ we have

$$\omega(u, Jv) = B(u, v) = B(v, u) = \omega(v, Ju)$$

while

$$\omega(u, JJv) = -\omega(u, v) = 0 = \omega(Jv, Ju)$$

Suppose that T preserves ω and B as above. We claim that

$$J_{TL} \circ T = T \circ J_L \tag{1}$$

so that T is complex linear for the complex structures J_L and J_{TL}. Indeed, for $u, v \in L$ we have

$$\omega(Tu, J_{TL}Tv) = B(Tu, Tv)$$

by the definition of J_{TL}. Since B is invariant under T the right hand side equals $B(u, v) = \omega(u, J_L v) = \omega(Tu, TJ_L v)$ since ω is invariant under T. Thus

$$\omega(Tu, J_{TL}Tv) = \omega(Tu, TJ_L v)$$

when applied to elements of L.

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showing that

$$TJ_L = J_{TL}T$$

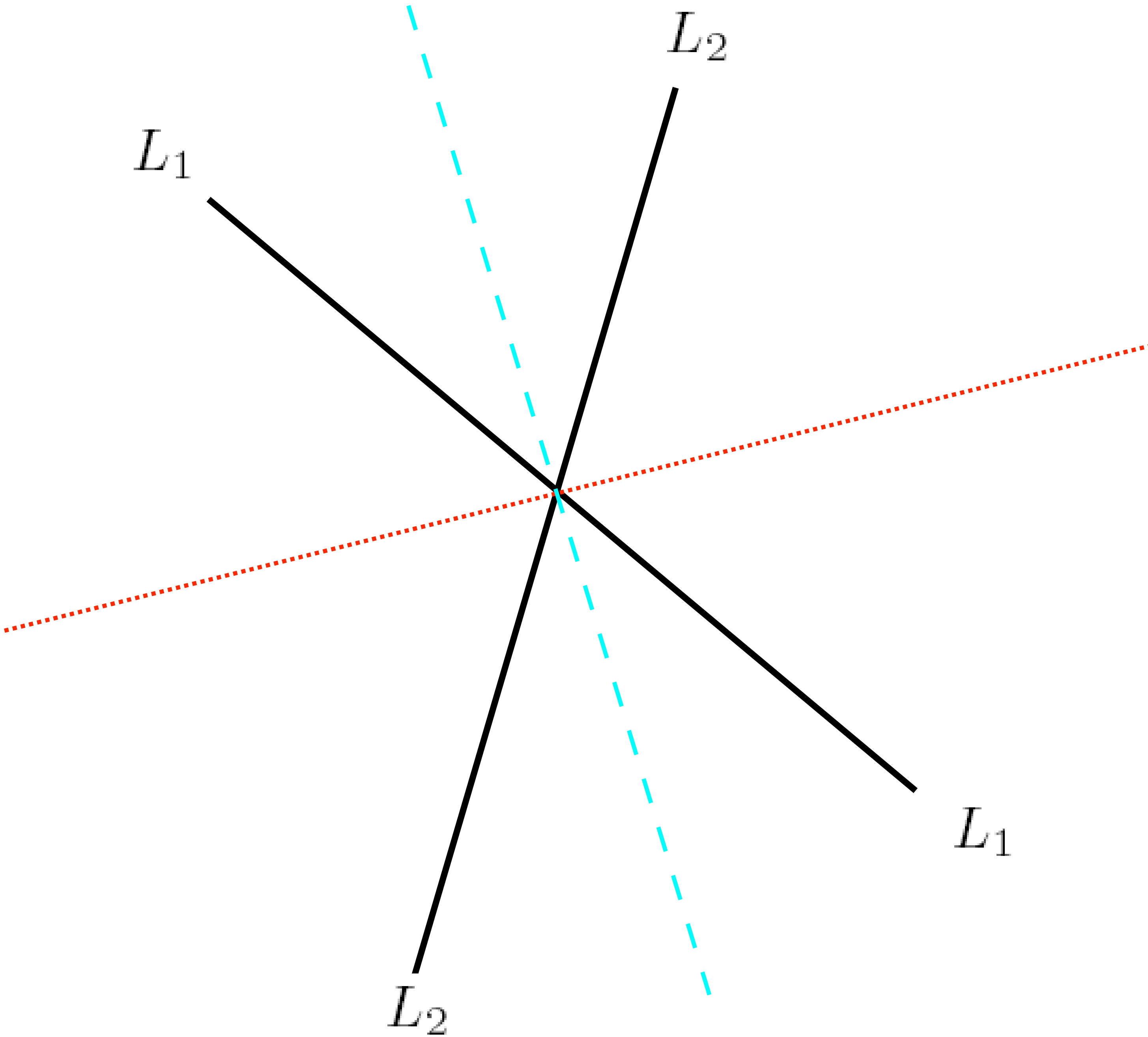
when applied to elements of L. This also holds for elements of L'. Indeed every element of L' is of the form $J_L u$ where $u \in L$ and $TJ_L u \in TL'$ so

$$J_{TL}TJ_L u = -J_{TL}^{-1}TJ_L u = -Tu = TJ_L(J_L u). \quad \square$$

The action of $Sp(V)$ on the space $\mathcal{L}(V)$.

Let $\mathcal{L}(V)$ denote the space of (real) Lagrangian subspaces of a symplectic vector space V. We know that $Sp(V)$ acts transitively on $\mathcal{L}(V)$. Suppose that L_1 and L_2 are two Lagrangian subspaces. An obvious invariant is the dimension of their intersection. Suppose that they are transverse, i.e. $L_1 \cap L_2 = \{0\}$. Any basis $e_1, \ldots, e_n$ of L_1 determines a dual basis $f_1, \ldots, f_n$ of L_2 and together they give a symplectic basis of V. Since $Sp(V)$ acts transitively on on the set of all symplectic bases, we see that $Sp(V)$ acts transitively on the set of all pairs of transverse Lagrangian subspaces.

But $Sp(V)$ does not act transitively on the space of all (ordered) triplets of Lagragian subspaces. We can see this already in the plane, where every line (through the origin) is a Lagrangian subspace. Once we fix two lines, the set of complementary lines is divided into two components corresponding to the two pairs of opposite cones complementary to the first to lines.


L_2
L_1
L_1
L_2

We can see this more analytically as follows: By an application of an element of $Sl(2) = Sp(V)$ we can arrange that L_1 is the x-axis and that L_2 is the y-axis. the subgroup of $Sl(2)$ which preserves these axes consists of all diagonal matrices

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

If $\lambda > 0$ such a matrix preserves all quadrants, and if $\lambda < 0$ it interchanges the first and third quadrant and the second and fourth quadrant. In any event such a matrix takes a line passing through the first and third quadrant into another such line, and the group of such matrices acts transitively on all such lines. Similarly for lines passing through the second and fourth quadrant.

The Maslov index.

So we want to study invariants of triplets of Lagrangian subspaces in a symplectic vector space.

Let L_1, L_2, L_3 be three (not necessarily transverse) Lagrangian subspaces of a symplectic vector space, V. On the $3n$ dimensional space

$$L_1 \oplus L_2 \oplus L_3$$

define the quadratic form

$$Q(x) = Q_{123}(x) :=$$

$$\omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1), \quad \text{for } x = x_1 \oplus x_2 \oplus x_3. \tag{2}$$

Define the **Maslov index**

$$\tau(L_1, L_2, L_3) := \text{sign } Q_{123}. \tag{3}$$

Recall that a quadratic form on a vectors space is always equivalent (under linear change or variables) to a sum of the form

$$\sum_{i=1}^{k} y_i^2 - \sum_{k+1}^{\ell} y_j^2.$$

The numbers k and ℓ are invariants of the quadratic form and $k-\ell$ is called its signature. In terms of a basis, any quadratic form is given by a symmetric matrix and the signature of the quadratic form is the number of positive eigenvalues of this matrix minus the number of negative eigenvalues.

$$Q_{123}(x) := \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1),$$

$$\tau(L_1, L_2, L_3) := \text{sign } Q_{123}.$$

Interchanging x_1 and x_2 in these definitions changes the sign of the first term and interchanges the last two terms with a change of sign. Similarly when we interchange x_2 and x_3. Hence

$$\tau(L_2, L_1, L_3) = -\tau(L_1, L_2, L_3), \quad \tau(L_1, L_3, L_2) = -\tau(L_1, L_2, L_3) \tag{4}$$

and hence

$$\tau(L_2, L_3, L_1) = \tau(L_1, L_2, L_3). \tag{5}$$

Also, it is clearly a symplectic invariant:

$$\tau(gL_1, gL_2, gL_3) = \tau(L_1, L_2, L_3), \quad \forall g \in Sp(V). \tag{6}$$

The Maslov index in terms of a basis.

Let us choose a Lagrangian subspace M transverse to L_1, L_2 and L_3, then a Lagrangian subspace L transverse to M and finally a basis of L. This gives a basis

$$g_1^{(1)}, \ldots, g_n^{(1)},\ g_1^{(2)}, \ldots, g_n^{(2)},\ g_1^{(3)}, \ldots, g_n^{(3)}$$

of $L_1 \oplus L_2 \oplus L_3$. In terms of this basis the symmetric matrix associated with Q takes the form

$$\begin{pmatrix} 0 & S_2 - S_1 & S_1 - S_3 \\ S_2 - S_1 & 0 & S_3 - S_2 \\ S_1 - S_3 & S_3 - S_2 & 0 \end{pmatrix}$$

where the S_i are the symmetric matricies parametrizing the L_i, $i = 1, 2, 3$.

Consider the matrix

$$T = \begin{pmatrix} 0 & I & I \\ I & 0 & I \\ 0 & I & I \end{pmatrix}.$$

Then $T = T^{\dagger}, \det T = 2^n$ and

$$TQT^{\dagger} = 2\begin{pmatrix} S_3 - S_2 & 0 & 0 \\ 0 & S_1 - S_3 & 0 \\ 0 & 0 & S_2 - S_1 \end{pmatrix}.$$

So in terms of this basis we have

$$\tau(L_1, L_2, L_3) = \mathrm{sig}(S_2 - S_1) + \mathrm{sig}(S_3 - S_2) + \mathrm{sig}(S_1 - S_3). \quad (7)$$

The cocycle identity.

We have proved:

$$\tau(L_1, L_2, L_3) = \mathrm{sig}(S_2 - S_1) + \mathrm{sig}(S_3 - S_2) + \mathrm{sig}(S_1 - S_3). \quad (7)$$

We claim that for any four Lagrangian subspaces we have

$$\tau(L_1, L_2, L_3) = \tau(L_1, L_2, L_4) + \tau(L_2, L_3, L_4) + \tau(L_3, L_1, L_4). \quad (8)$$

Indeed, by (7) we have the local expressions

$$\begin{aligned}
\tau(L_1, L_2, L_4) &= \mathrm{sig}(S_2 - S_1) + \mathrm{sig}(S_4 - S_2) + \mathrm{sig}(S_1 - S_4) \\
\tau(L_2, L_3, L_4) &= \mathrm{sig}(S_3 - S_2) + \mathrm{sig}(S_4 - S_3) + \mathrm{sig}(S_2 - S_4) \\
\tau(L_3, L_1, L_4) &= \mathrm{sig}(S_1 - S_3) + \mathrm{sig}(S_4 - S_1) + \mathrm{sig}(S_3 - S_4).
\end{aligned}$$

Adding and using (7) again proves (8).

Proposition 6 *Let $L_1(t), L_2(t), L_3(t)$ be a triplet of continuous curves of Lagrangian subspaces and let*

$$\tau(t) := \tau(L_1(t), L_2(t), L_3(t)).$$

Suppose that the dimensions

$$\dim(L_1(t) \cap L_2(t)), \quad \dim(L_2(t), L_3(t)), \; \dim(L_3(t) \cap L_1(t))$$

are all constant. Then $\tau(t)$ is constant. Furthermore

$$\tau(t) \equiv$$

$$n + \dim(L_1(t) \cap L_2(t)) + \dim(L_2(t), L_3(t)) + \dim(L_3(t) \cap L_1(t)) \bmod 2.$$

Proof. In the local expression (7) the nullity of the quadratic form $S_2 - S_1$, for example, is just $\dim(L_1 \cap L_2)$ and similarly for the other two terms. So if these dimensions don't change, the rank of the quadratic form Q does not change, and hence its signature does not change. This proves the first part. For the second assertion, we have

$$\tau(L_1, L_2, L_3) = p - q$$

where p is the number of positive summands and q the number of negative summands in the diagonalization of Q. So $p + q =$ rank $(Q) = 3n -$nullity Q and we have just computed this nullity to be the sum of the dimensions in question. $\square$

In particular, suppose that L_1 and L_2 are fixed and that $M(t)$ is a contiunous path of Lagrangian subspaces which are all transversal to L_1 and L_2. Then $\tau(L_1, L_2, M(t))$ is a constant.

The Maslov intersection index.

Lemma 1 *Let $L_1(t)$ and $L_2(t)$ be two continuous paths of Lagrangian subspaces $a \leq t \leq b$. Suppose that there is an M transversal to all $L_1(t)$ and all $L_2(t)$ for all $a \leq t \leq b$. Then*

$$[L_1 : L_2]_{[a,b]} := \frac{1}{2}\left[\tau(L_1(a), L_2(a), M) - \tau(L_1(b), L_2(b), M)\right]$$

is independent of the choice of M.

Proof. Suppose M and M' are two such choices. By the Proposition, $\tau(L_1(t), M, M')$ and $\tau(L_2(t), M, M')$ are independent of t. But by the cocycle identity (8)

$$\begin{aligned} \tau(L_1(a), L_2(a), M) - \tau(L_1(a), L_2(a), M) &= \tau(L_1(a), M, M') - \tau(L_2(a), M, M') \\ \tau(L_1(b), L_2(b), M) - \tau(L_1(b), L_2(b), M) &= \tau(L_1(b), M, M') - \tau(L_2(b), M, M'). \quad \square \end{aligned}$$

We define the Maslov intersection index for two arbitrary continuous paths $L_1, L_2 : [a,b] \to \mathcal{L}(V)$ as follows: Choose a subdivision $a = t_0 < t_1 < \cdots < t_n = b$ such that in each interval $[t_i, t_{i+1}]$ there is an $M_i \in \mathcal{L}(V)$ transverse to all the $L_1(t)$ and $L_2(t)$ for $t \in [t_i, t_{i+1}]$. Then define

$$[L_1, L_2] := \sum [L_1 : L_2]_{[t_i, t_{i+1}]}.$$

This is clearly independent of the choice of subdivision.

Of particular importance is the invariant

$$e^{\frac{\pi i}{4}\tau(L_1,L_2,L_3)}$$

associating to every triplet of Lagrangian subspaces an eighth root of unity. This invariant is very important in the theory of Fourier integral operators.

$$e^{\frac{\pi i}{4}\tau(L_1,L_2,L_3)}$$

For the rest of today's lecture I will try to describe (without proofs) how this invariant is a boundary limit of a beautiful invariant associated to three points in the interior of a certain complex domain - the Siegel domain **D** consisting of the positive definite Lagrangian subspaces of the complexification of V. That is, we will identify $\mathcal{L}(V)$ as the so-called Shilov boundary of $\mathcal{D}$, and define an invariant (taking values in the unit circle) to three points in the interior of **D** so that when passing to the boundary component $\mathcal{L}(V)$ the invariant becomes $e^{\frac{\pi i}{4}\tau(L_1,L_2,L_3)}$. To do this, I need first of all to state the Stone- von Neumann theorem from the foundations of quantum mechanics. This theorem asserts the uniqueness of the "representation of the Heisenberg commutation relations in Weyl form". For a precise statement see Theorem 1 below.

The Heisenberg algebra and Group.

Let V be a symplectic vector space. So V comes equipped with a skew symmetric non-degenerate bilinear form ω. By the choice of a pair of transverse Lagrangian subspaces, and then dual bases in these subspaces, we obtain a basis

$$P_1, \ldots, P_n, Q_1, \ldots, Q_n$$

of V with

$$\begin{aligned} \omega(P_i, P_j) &= 0 \\ \omega(Q_i, Q_j) &= 0 \\ \omega(P_i, Q_j) &= \delta_{ij}. \end{aligned} \tag{9}$$

We make

$$\mathbf{h} := V \oplus R$$

into a Lie algebra by defining

$$[X, Y] := \omega(X, Y)E$$

where $E = 1 \in \mathbf{R}$ and

$$[E, E] = 0 = [E, X] \quad \forall X \in V.$$

The Lie algebra $\mathbf{h}$ is called the **Heisenberg algebra**. It is a nilpotent Lie algebra. In fact, the Lie bracket of any three elements is zero. If we write out the brackets in terms of the basis above we get

$$\begin{aligned}
[P_i, Q_j] &= \delta_{ij} E \\
[P_i, P_j] &= 0 \\
[Q_i, Q_j] &= 0
\end{aligned}$$

The “canonical commutation” relations.

$$\begin{aligned}[P_i, Q_j] &= \delta_{ij} E \\ [P_i, P_j] &= 0 \\ [Q_i, Q_j] &= 0\end{aligned}$$

which, together with

$$[E, P_j] = 0 = [E, Q_j]$$

are the “canonical commutation relations” up to inessential (or essential) factors such as $\hbar$ and i.

The Heisenberg group.

We will let N denote the simply connected Lie group with this Lie algebra. We may identify the $2n+1$ dimensional vector space $V + \mathbf{R}$ with N via the exponential map, and with this identification the multiplication law on N reads

$$\exp(v+tE)\exp(v'+t'E) = \exp\left(v + v' + (t + t' + \frac{1}{2}\omega(v, v'))E\right). \tag{10}$$

Let dv be the Euclidean (Lebesgue) measure on V. Then the measure $dvdt$ is invariant under left and right multiplication.

If ℓ is a Lagrangian subspace of V, then $\ell \oplus \mathbf{R}$ is an Abelian subalgebra of $\mathbf{h}$, and in fact is maximal abelian. Similarly

$$L := \exp(\ell \oplus \mathbf{R})$$

is a maximal Abelian subgroup of N.

Define the function

$$f : N \to T^1$$

$$f(\exp(v + tE)) := e^{2\pi i t}.$$

We have

$$f\left((\exp(v + tE))(\exp(v' + t'E))\right) = e^{2\pi i (t + t' + \frac{1}{2}\omega(v,v'))}. \qquad (11)$$

Therefore

$$f(h_1 h_2) = f(h_1) f(h_2) \qquad h_1, \quad h_2 \in L.$$

I want to consider the quotient space

$$N/L$$

which has a natural action of N (via left multiplication). In other words N/L is a homogeneous space for the Heisenberg group N. Let ℓ' be a Lagrangian subspace transverse to ℓ. Every element of N has a unique expression as

$$(\exp y)(\exp(x + sE)) \quad \text{where } y \in \ell' \quad x \in \ell.$$

This allows us to make the identification

$$N/L \sim \ell'$$

and the Euclidean measure dv' on ℓ' then becomes identified with the (unique up to scalar multiple) measure on N/L invariant under N.

For use in the next section we record the following "commutation calculation" at the group level: Let $y \in \ell'$ and $x \in \ell$. Then

$$\exp(-x)(\exp y) = \exp(y - x - \frac{1}{2}\omega(x, y)E)$$

while

$$\exp(y)\exp(-x) = \exp(y - x - \frac{1}{2}\omega(y, x)E)$$

so, since ω is antisymmetric, we get

$$(\exp(-x))(\exp y) = (\exp y)(\exp(-x))\exp(-\omega(x, y)E). \quad (12)$$

The Schrodinger representation.

We continue with the notation of the preceding section. In particular, we have chosen a Lagrangian subspace ℓ, have the corresponding subgroup L and the quotient space N/L. We are going to construct a unitary representation of N which is known in group theory language as the representation of N induced from the character f of L.

Its definition is as follows: Consider the space of continuous functions ϕ on N which satisfy

$$\phi(nh) = f(h)^{-1}\phi(n) \quad \forall\ n \in N \quad h \in L \tag{13}$$

and which in addition have the property that the function

$$n \mapsto |\phi(n)|$$

(which is well defined on N/L on account of (13)) is square integrable.

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(which is well defined on N/L on account of (13)) is square integrable. We let $H(\ell)$ denote the Hilbert space which is the completion of this space of continuous functions relative to this L_2 norm. So $\phi \in H(\ell)$ is a "function" on N satisfying (13) with norm

$$\|\phi\|^2 = \int_{N/L} |\phi|^2 d\dot{n}$$

where $d\dot{n}$ is left invariant measure on N/L.

The representation ρ_ℓ of N on $H(\ell)$ is given by left translation:

$$(\rho_\ell(m)\phi)(n) := \phi\left(m^{-1}n\right). \tag{14}$$

For the rest of this section we will keep ℓ fixed, and so may write H for $H(\ell)$ and ρ for ρ_ℓ. The dependence on ℓ will become important for us later.

Since $\exp tE$ is in the center of N, we have

$$\rho(\exp tE)\phi(n) = \phi\left((\exp -tE)n\right) = \phi\left(n(\exp -tE)\right) = e^{2\pi it}\phi(n).$$

In other words

$$\rho\left(\exp tE\right) = e^{2\pi it}\mathrm{Id}_H. \tag{15}$$

The Stone - von Neumann (Theorem 1 below) characterizes all unitary representations of N which satisfy this condition.

$$\phi(nh) = f(h)^{-1}\phi(n) \quad \forall\, n \in N \quad h \in L \tag{13}$$

$$(\rho_\ell(m)\phi)(n) := \phi\left(m^{-1}n\right). \tag{14}$$

Suppose we choose a complementary Lagrangian subspace ℓ' and then identify N/L with ℓ' as in the preceding section. Condition (13) becomes

$$\phi\left((\exp y)(\exp(x))(\exp tE)\right) = \phi(\exp y)e^{-2\pi i t}.$$

So $\phi \in H$ is completely determined by its restriction to $\exp \ell'$. In other words the map

$$\phi \mapsto \psi, \quad \psi(y) := \phi(\exp y)$$

defines a unitary isomorphism

$$R : H \to L_2(\ell')$$

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and if we set

$$\sigma := R\rho R^{-1}$$

then

$$\begin{aligned}
[\sigma\,(\exp x)\,\psi](y) &= e^{2\pi i\omega(x,y)}\psi(y) \quad & x \in \ell,\ y \in \ell' \\
[\sigma(\exp u)\psi](y) &= \psi(y-u) & y, u \in \ell' \\
\sigma(\exp(tE)) &= e^{2\pi it}\mathrm{Id}_{L_2}(\ell')
\end{aligned} \tag{16}$$

$$\begin{array}{rcll} [\sigma(\exp x)\psi](y) & = & e^{2\pi i\omega(x,y)}\psi(y) & x\in\ell,\ y\in\ell' \\ [\sigma(\exp u)\psi](y) & = & \psi(y-u) & y,u\in\ell' \\ \sigma(\exp(tE)) & = & e^{2\pi it}\mathrm{Id}_{L_2}(\ell') & \end{array} \tag{16}$$

We define the infinitesimal version of the representation ρ by

$$\dot{\rho}(X) := \frac{d}{dt}\rho(\exp(tX))_{|t=0}$$

for $X \in \mathbf{h}$ with a similar notion and notation for σ. Under the P, Q basis (with $P_i \in \ell$ chosen above), we may identify $L_2(\ell')$ with $L_2(\mathbf{R}^n)$. Then it follows from (16) that

$$\begin{array}{rcl} \dot{\sigma}(P_j) & = & 2\pi i x_j \\ \dot{\sigma}(Q_j) & = & -\frac{\partial}{\partial x_j} \\ \dot{\sigma}(E) & = & 2\pi i \mathrm{Id} \end{array} \tag{17}$$

$$\begin{array}{rclr} [\sigma\,(\exp x)\,\psi](y) & = & e^{2\pi i\omega(x,y)}\psi(y) \quad x\in\ell,\ y\in\ell' & \\ [\sigma(\exp u)\psi](y) & = & \psi(y-u) \qquad\qquad y,u\in\ell' & (16) \\ \sigma(\exp(tE)) & = & e^{2\pi it}\mathrm{Id}_{L_2}(\ell') & \end{array}$$

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$$\begin{array}{rcl} \dot\sigma(P_j) & = & 2\pi i x_j \\ \dot\sigma(Q_j) & = & -\frac{\partial}{\partial x_j} \\ \dot\sigma(E) & = & 2\pi i\mathrm{Id} \end{array} \qquad (17)$$

This is the Schrodinger version of the Heisenberg commutation relations. So we can regard (16) as an "integrated version" of the Heisenberg commutation relations. The Stone-von Neumann theorem asserts that the representation σ, and hence the representation ρ is irreducible and is the unique irreducible representation (up to isomorphism) satisfying (16).

Let H_1 and H_2 be Hilbert spaces. We can form their tensor product as vector spaces, and this tensor product inherits a scalar product determined by

$$(u \otimes v, x \otimes y) = (u, x)(v, y).$$

The completion of this (algebraic) tensor product with respect to this scalar product will be denoted by $H_1 \otimes H_2$ and will be called the (Hilbert space) tensor product of H_1 and H_2. If we have a representation τ of a group G on H_1 we get a representation

$$g \mapsto \tau(g) \otimes \mathrm{Id}_{H_2}$$

on $H_1 \otimes H_2$ which we call a multiple of the representation τ. We can now state:

Theorem 1 **[The Stone-von-Neumann theorem.]** *The representation* $\rho(\ell)$ *of* N *is irreducible, and any representation such that* $\exp(tE) \mapsto e^{2\pi i t}\mathrm{Id}$ *is isomorphic to a multiple of* $\rho(\ell)$.

This theorem was first conjectured by Hermann Weyl and then proved by Stone and von Neumann.

A description of the metaplectic representation.

Let V be a symplectic vector space, $\mathbf{h}$ the associated Heisenberg algebra, and N the corresponding Heisenberg group. Let τ be any irreducible representation satisfying $\tau(\exp tE) = e^{2\pi it}I$. By the Stone - von Neumann theorem we know that τ is unique up to a unitary equivalence. Any $M \in Sp(V)$ acts as an automorphism of $\mathbf{h}$, hence as an automporphism of N preserving the center, which we will continue to denote by M. Hence τ_M defined by

$$\tau_M(n) = \tau(Mn)$$

is another irreducible representation of N satisfying the same condition on the center. Hence there is a unitary map U_M detemined up to multiplication by a scalar of absolute value 1 such that

$$\tau_M(n) = U_M \tau(a) U_M^{-1}.$$

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Then it follows that

$$U_{M_1} U_{M_2} = c(M_1, M_2) U_{M_1 M_2}.$$

In other words the map $M \mapsto U_M$ is what is known as a projective representation of the group $Sp(V)$ with cocycle c. By general principles of group theory, this implies that this corresponds to an honest unitary representation of the universal cover of $Sp(V)$. In fact it is a representation of the double cover, and you have enough information to prove this fact. Indeed, if you take the τ to be Schrodinger realization ρ that we just constructed, and use the operators U you developped in the homework, you can check that if U corresponds to the matrix M (remember that this was two to one) then

$$U \rho(a) U^{-1} = \rho(Ma).$$

Indeed, you only have to verify this for the operators U_d and V_P since they generate, and this is a direct verification.

We will spend a lot of time in the notes giving another verification of this fact and identifying the cocycle as being related to the square root of a certain determinant - the need to pass to the double cover coming from the two choices in the sign of the square root.

We will also give many other realizations of τ. Instead of constructing τ from a real Lagrangian subspace, we will use certain complex Lagrangian subspaces. They have the advantage that they possess a unique "vacuum state". What I mean is this. Suppose we look for an element in the representation space (say in the Schrodinger realization - remember that all are equivalent) which is annhilated by all the $\dot{\tau}(Q_j)$. If these are realized by

$$\frac{\partial}{\partial x_j},$$

then the (one dimensional space of) constants are the only guys annihilated by all the Q_j. They do not lie in the Hilbert space.

Similarly, as the P_j are realized as multiplication by $2\pi i x_j$ and only elements annihilated by all these are (multiples of) the delta function which again does not belong to the Hilbert space. But suppose we consider the element annihilated by the $Q_j - iP_j$. This will be (all multiples of) a Gaussian, which *does* belong to the Hilbert space. So by passing from real to certain complex Lagrangian subspaces, we get a canonical line lying in the Stone von Neumann representation.

We will begin by studying the space of "positive complex Lagrangian subspaces", and see that they form a natural generalization to $2n$ dimensions of the unit disk in the complex plane. We will associate to each such point in the "generalized unit disk" a tiny subspace of a certain huge Hilbert space, which will be the realization corresponding to this point of the Stone - von Neumann representation. Each of these subspaces has a unique line of "vacuum vectors", and no two of these lines are orthogonal in the huge Hilbert space. Given any three non-orthogonal lines in a Hilbert space there is an associated invariant:

$$\arg\,(v_1, v_2)(v_2, v_3)(v_3, v_1), \quad 0 \neq v_i \in \ell_i,\ i = 1, 2, 3.$$

(The left hand side does not depend on the choice of the v_i.) In fact, although we will not prove it, this number is essentially e^{iA} where A is the area of the triangle spanned by the three points in the hyperbolic geometry of the unit disk.)

It is this invariant $\arg\,(v_1, v_2)(v_2, v_3)(v_3, v_1), \quad 0 \neq v_i \in \ell_i,\ i = 1, 2, 3$ which becomes $e^{\frac{\pi i}{4}\tau(L_1, L_2, L_3)}$ as the $\ell_i \to L_i$ on the boundary. For details see the notes.