Symplectic Geometry
Lecture 3.

The linear symplectic category.

The purpose of today's lecture is to describe the category whose objects are symplectic vector spaces and whose morphisms are the linear canonical relations. Everything in these this lecture reflects joint work with Victor Guillemin.

References for today:

V. Guillemin and S. Sternberg, "Some problems in integral geometry and some related problems in microlocal analysis", Am. J. Math., Vol 101 (1979) 915–955.

Alan Weinstein, "Symplectic geometry" Bull. Amer. Math. Soc. 5 (1981) 1–13.

The language of category theory.

A **category C** consists of the following data:

(i) A set, $Ob(\mathbf{C})$, whose elements are called the **objects** of **C** ,

(ii) For every pair (X, Y) of $Ob(\mathbf{C})$ a set, $\mathrm{Morph}(X, Y)$, whose elements are called the **morphisms** or **arrows** from X to Y,

(iii) For every triple (X, Y, Z) of $Ob(\mathbf{C})$ a map from $\mathrm{Morph}(X, Y) \times \mathrm{Morph}(Y, Z)$ to $\mathrm{Morph}(X, Z)$ called the **composition map** and denoted $(f, g) \rightsquigarrow g \circ f$.

The axioms for a category:

These data are subject to the following conditions:

(iv) The composition of morphisms is *associative*

(v) For each $X \in Ob(\mathbf{C})$ there is an $id_X \in \mathrm{Morph}(X, X)$ such that

$$f \circ id_X = f, \ \forall f \in \mathrm{Morph}(X, Y)$$

(for any Y) and

$$id_X \circ f = f, \ \forall f \in \mathrm{Morph}(Y, X)$$

(for any Y).

It follows from the definitions that id_X is unique.

Functors and morphisms.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, a **functor** F from $\mathcal{C}$ to $\mathcal{D}$ consists of the following data:

(vi) a map $F : Ob(\mathcal{C}) \to Ob(\mathcal{D})$

and

(vii) for each pair (X, Y) of $Ob(\mathcal{C})$ a map

$$F : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$$

subject to the rules

(viii)

$$F(id_X) = id_{F(X)}$$

and

(ix)

$$F(g \circ f) = F(g) \circ F(f).$$

This is what is usually called a **covariant functor**.

A **contravariant functor** would have $F : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(Y), F(X))$ in (vii) and $F(f) \circ F(g)$ on the right hand side of (ix).)

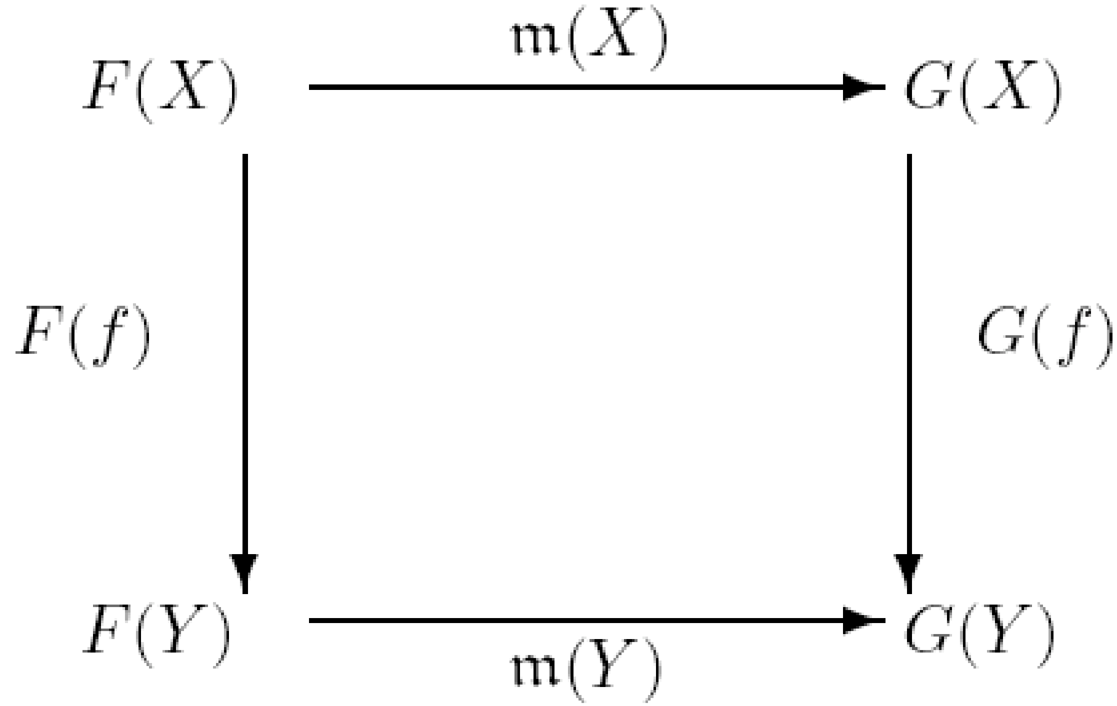

Figure 3.1:

Let F and G be two functors from $\mathcal{C}$ to $\mathcal{D}$. A **morphism**, $\mathfrak{m}$, from F to G (older name: "natural transformation") consists of the following data:

(x) for each $X \in Ob(\mathcal{C})$ an element $\mathfrak{m}(X) \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ subject to the "naturality condition"

(xi) for any $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ the diagram in Figure 3.1 commutes.

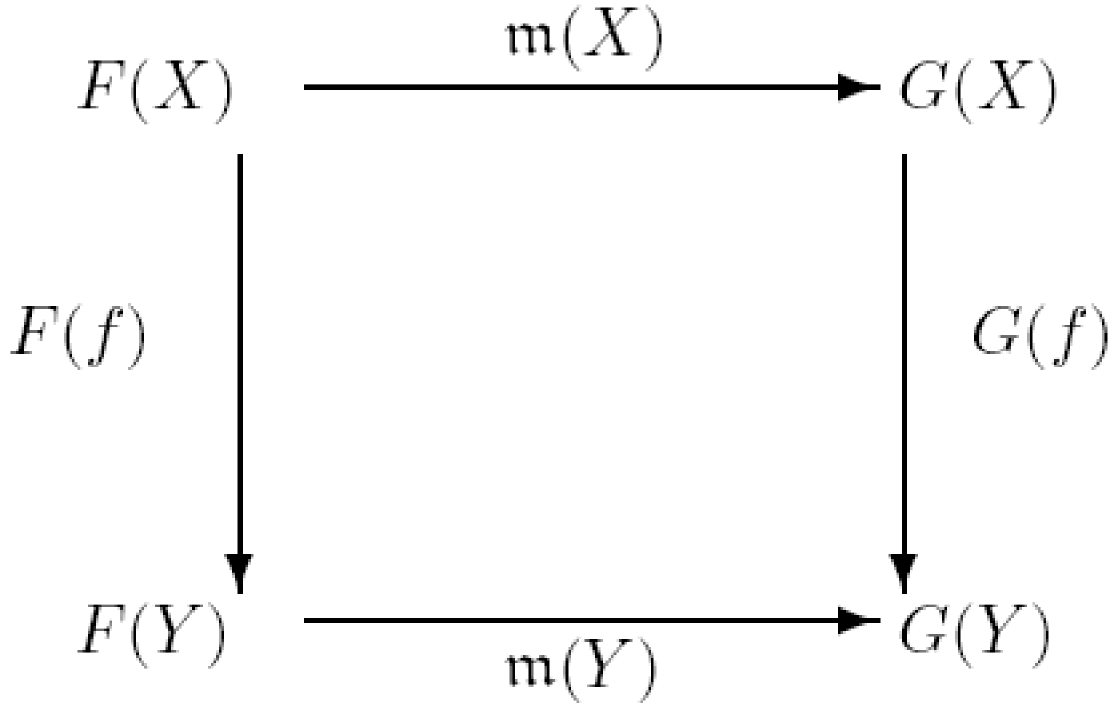

Figure 3.1:

(x) for each $X \in Ob(\mathcal{C})$ an element $\mathfrak{m}(X) \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X))$ subject to the "naturality condition"

(xi) for any $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ the diagram in Figure 3.1 commutes. In other words

$$\mathfrak{m}(Y) \circ F(f) = G(f) \circ \mathfrak{m}(X) \quad \forall\, f \in f \in \mathrm{Hom}_{\mathcal{C}}(X,Y).$$

Involutory functors.

Consider the category $\mathcal{V}$ whose objects are finite dimensional vector spaces (over some given field $\mathbb{K}$) and whose morphisms are linear transformations. We can consider the "transpose functor" $F : \mathcal{V} \to \mathcal{V}$ which assigns to every vector space V its dual space

$$V^* = \mathrm{Hom}(V, \mathbb{K})$$

and which assigns to every linear transformation $\ell : V \to W$ its transpose

$$\ell^* : V^* \to W^*.$$

In other words,

$$F(V) = V^*, \quad F(\ell) = \ell^*.$$

This is a contravariant functor which has the property that F^2 is naturally equivalent to the identity functor. There does not seem to be a standard name for this type of functor. We will call it an **involutory** functor.

Involutive functors.

A special type of involutory functor is one in which $F(X) = X$ for all objects X and $F^2 = \mathrm{id}$ (not merely naturally equivalent to the identity). For example, let $\mathcal{H}$ denote the category whose objects are Hilbert spaces and whose morphisms are bounded linear transformations. We take $F(X) = X$ on objects and $F(L) = L^\dagger$ on maps where $L^\dagger$ denotes the adjoint of L in the Hilbert space sense. We shall call such a functor a **involutive** functor. We will refer to a category with an involutive functor as an involutive category, or say that we have a category with an involutive structure.

Sets, maps and relations.

The category **Set** is the category whose objects are ("all") sets and and whose morphisms are ("all") maps between sets. For reasons of logic, the word "all" must be suitably restricted to avoid contradiction.

We will take the extreme step in this section of restricting our attention to the class of finite sets. Our main point is to examine a category whose objects are finite sets, but whose morphisms are much more general than maps. Some of the arguments and constructions that we use in the study of this example will be models for arguments we will use later on, in the context of the symplectic "category".

The category of finite relations.

We will consider the category whose objects are finite sets. But we enlarge the set of morphisms by defining

$\mathrm{Morph}(X, Y) =$ the collection of all subsets of $X \times Y$.

A subset of $X \times Y$ is called a **relation**. We must describe the map

$$\mathrm{Morph}(X, Y) \times \mathrm{Morph}(Y, Z) \to \mathrm{Morph}(X, Z)$$

and show that this composition law satisfies the axioms of a category.

So let

$$\Gamma_1 \in \mathrm{Morph}(X, Y) \quad \text{and } \Gamma_2 \in \mathrm{Morph}(Y, Z).$$

Define

$$\Gamma_2 \circ \Gamma_1 \subset X \times Z$$

by

$$(x, z) \in \Gamma_2 \circ \Gamma_1 \;\Leftrightarrow\; \exists\, y \in Y \text{ such that } (z, y) \in \Gamma_1 \text{ and } (y, z) \in \Gamma_2. \qquad (3.1)$$

Notice that if $f : X \to Y$ and $g : Y \to Z$ are maps, then

$$\operatorname{graph}(f) = \{(x, f(x)\} \in \operatorname{Morph}(X, Y) \quad \text{and}$$

$$\operatorname{graph}(g) \in \operatorname{Morph}(Y, Z)$$

with

$$\operatorname{graph}(g) \circ \operatorname{graph}(f) = \operatorname{graph}(g \circ f).$$

So we have indeed enlarged the category of finite sets and maps.

We still must check the axioms. Let $\Delta_X \subset X \times X$ denote the diagonal:

$$\Delta_X = \{(x, x),\ x \in X\},$$

so

$$\Delta_X \in \mathrm{Morph}(X, Y).$$

If $\Gamma \in \mathrm{Morph}(X, Y)$ then

$$\Gamma \circ \Delta_X = \Gamma \quad \text{and } \Delta_Y \circ \Gamma = \Gamma.$$

So Δ_X satisfies the conditions for id_X.

The associative law.

Suppose that $\Gamma_1 \in \text{Morph}(X, Y), \Gamma_2 \in \text{Morph}(Y, Z)$ and

$\Gamma_3 \in \text{Morph}(Z, W)$. Then both

$\Gamma_3 \circ (\Gamma_2 \circ \Gamma_1)$ and $(\Gamma_3 \circ \Gamma_2) \circ \Gamma_1$ consist of all $(x, w) \in X \times W$ such that there exist $y \in Y$ and $z \in Z$ with

$$(x, y) \in \Gamma_1, \ (y, z) \in \Gamma_2, \quad \text{and } (z, w) \in \Gamma_3.$$

This proves the associative law.

Let us call this category **FinRel**.

Categorical "points".

Let us pick a distinguished one element set and call it "pt.". Giving a *map* from pt. to any set X is the same as picking a point of X. So in the category **Set** of sets and *maps*, the points of X are the same as the morphisms from our distinguished object pt. to X.

In a more general category, where the objects are not necessarily sets, we can not talk about the points of an object X. However if we have a distinguished object pt., then we can *define* a **"point"** of any object X to be an element of $\mathrm{Morph}(\mathrm{pt.}, X)$. For example, later on, when we study the symplectic "category" whose objects are symplectic manifolds, we will find that the "points" in a symplectic manifold are its Lagrangian submanifolds. This idea has been emphasized by Weinstein.

“Points” of **FinRel**.

In the category **FinRel**, the category of finite sets and relations, an element of $\mathrm{Morph}(\mathrm{pt.}, X)$, i.e a subset of $\mathrm{pt.} \times X$ is the same as a subset of X (by projection onto the second factor). So in this category, the “points” of X are the subsets of X. Many of the constructions we do here can be considered as warm ups to similar constructions in the symplectic “category”.

A morphism $\Gamma \in \mathrm{Morph}(X, Y)$ yields a map from “points” of X to “points” of Y.

The twisted diagonal.

Consider the following example: For three objects X, Y, Z in

$$X \times X \times Y \times Y \times Z \times Z$$

we have the subset

$$\Delta_X \times \Delta_Y \times \Delta_Z .$$

Let us move the first X factor past the others until it lies to immediate left of the right Z factor, so consider the subset

$$\tilde{\Delta}_{X,Y,Z} \subset X \times Y \times Y \times Z \times X \times Z, \quad \tilde{\Delta}_{X,Y,Z} = \{(x, y, y, z, x, z)\}.$$

$$\tilde{\Delta}_{X,Y,Z} \subset X \times Y \times Y \times Z \times X \times Z, \quad \tilde{\Delta}_{X,Y,Z} = \{(x,y,y,z,x,z)\}.$$

By introducing parentheses around the first four and last two factors we can write

$$\tilde{\Delta}_{X,Y,Z} \subset (X \times Y \times Y \times Z) \times (X \times Z).$$

In other words,

$$\tilde{\Delta}_{X,Y,Z} \in \mathrm{Morph}(X \times Y \times Y \times Z, X \times Z).$$

Let $\Gamma_1 \in \mathrm{Morph}(X, Y)$ and $\Gamma_2 \in \mathrm{Morph}(Y, Z)$. Then

$$\Gamma_1 \times \Gamma_2 \subset X \times Y \times Y \times Z$$

is a "point" of $X \times Y \times Y \times Z$. We identify this "point" with an element of

$$\mathrm{Morph}(\mathrm{pt.}, X \times Y \times Y \times Z)\ .$$

$$\tilde{\Delta}_{X,Y,Z} \subset X \times Y \times Y \times Z \times X \times Z, \quad \tilde{\Delta}_{X,Y,Z} = \{(x, y, y, z, x, z)\}.$$

$$\tilde{\Delta}_{X,Y,Z} \in \mathrm{Morph}(X \times Y \times Y \times Z, X \times Z).$$

$$\Gamma_1 \times \Gamma_2 \subset X \times Y \times Y \times Z$$

is a "point" of $X \times Y \times Y \times Z$. We identify this "point" with an element of

$$\mathrm{Morph}(\mathrm{pt.}, X \times Y \times Y \times Z)$$

so that we can form

$$\tilde{\Delta}_{X,Y,Z} \circ (\Gamma_1 \times \Gamma_2)$$

$$\tilde{\Delta}_{X,Y,Z} = \{(x, y, y, z, x, z)\}.$$

$$\tilde{\Delta}_{X,Y,Z} \circ (\Gamma_1 \times \Gamma_2)$$

consists of all (x, z) such that

$\exists(x_1, y_1, y_2, z_1, x, z)$ with

$$\begin{aligned} (x_1, y_1) &\in \Gamma_1, \\ (y_2, z_1) &\in \Gamma_2 \\ x_1 &= x \\ y_1 &= y_2 \\ z_1 &= z. \end{aligned}$$

Thus

$$\tilde{\Delta}_{X,Y,Z} \circ (\Gamma_1 \times \Gamma_2) = \Gamma_2 \circ \Gamma_1. \qquad (3.2)$$

Similarly, given four sets X, Y, Z, W we can form

$$\tilde{\Delta}_{X,Y,Z,W} \subset (X \times Y \times Y \times Z \times Z \times W) \times (X \times W)$$

$$\tilde{\Delta}_{X,Y,Z,W} = \{(x, y, y, z, z, w, x, w)\}$$

so

$$\tilde{\Delta}_{X,Y,Z,W} \in \mathrm{Morph}(X \times Y \times Y \times Z \times Z \times W, X \times W).$$

If $\Gamma_1 \in \mathrm{Morph}(X, Y)$, $\Gamma_2 \in \mathrm{Morph}(Y, Z)$, and $\Gamma_3 \in \mathrm{Morph}(Z, W)$ then

$$\Gamma_3 \circ (\Gamma_2 \circ \Gamma_1) = (\Gamma_3 \circ \Gamma_2) \circ \Gamma_1 = \tilde{\Delta}_{X,Y,Z,W}(\Gamma_1 \times \Gamma_2 \times \Gamma_3).$$

The associative law via "points".

From this point of view the associative law is a reflection of the fact that

$$(\Gamma_1 \times \Gamma_2) \times \Gamma_3 = \Gamma_1 \times (\Gamma_2 \times \Gamma_3) = \Gamma_1 \times \Gamma_2 \times \Gamma_3.$$

The involutive structure on **FinRel**.

In our category **FinRel**, if $\Gamma \in \text{Morph}(X, Y)$ define $\Gamma^\dagger \in \text{Morph}(Y, X)$ by

$$\Gamma^\dagger := \{(y, x) | (x, y) \in \Gamma\}.$$

We have defined a map

$$\dagger : \text{Morph}(X, Y) \to \text{Morph}(Y, X) \tag{3.3}$$

for all objects X and Y which clearly satisfies

$$\dagger^2 = \text{id} \tag{3.4}$$

and

$$(\Gamma_2 \circ \Gamma_1)^\dagger = \Gamma_1^\dagger \circ \Gamma_2^\dagger. \tag{3.5}$$

The finite Radon transform.

This is a contravariant functor $\mathcal{F}$ from the category **FinRel** to the category of finite dimensional vector spaces over a field $\mathbb{K}$. It is defined as follows: On objects we let

$$\mathcal{F}(X) := \mathcal{F}(X, \mathbb{K}) = \text{the space of all } \mathbb{K}\text{-valued functions on } X$$

If $\Gamma \subset X \times Y$ is a relation and $g \in \mathcal{F}(Y)$ we set

$$(\mathcal{F}(\Gamma)(g))(x) := \sum_{y|(x,y)\in\Gamma} g(y) \ \ \forall\, x \in X.$$

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$$(\mathcal{F}(\Gamma)(g))(x) := \sum_{y|(x,y)\in\Gamma} g(y) \ \ \forall\ x \in X.$$

(It is understood that the empty sum gives zero.) It is immediate to check that this is indeed a contravariant functor.

In case $\mathbb{K} = \mathbb{C}$ we can be more precise: Let us make $\mathcal{F}(X)$ into a (finite dimensional) Hilbert space by setting

$$(f_1, f_2) := \sum_{x \in X} f_1(x)\overline{f_2(x)}.$$

Then for $\Gamma \in \text{Morph}(X, Y), f \in \mathcal{F}(X), g \in \mathcal{F}(Y)$ we have

$$(f, \mathcal{F}(\Gamma)g) = \sum_{(x,y) \in \Gamma} f(x)\overline{g(y)} = (\mathcal{F}(\Gamma^\dagger)f, g).$$

So

$$\mathcal{F}(\Gamma^\dagger) = \mathcal{F}(\Gamma)^\dagger,$$

i.e.

$$\mathcal{F} \circ \dagger = \dagger \circ \mathcal{F}$$

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where the $\dagger$ on the left is the involution on **FinRel** and the $\dagger$ on the right is the operation carrying a linear transformation between Hilbert spaces into its adjoint. Thus the functor $\mathcal{F}$ carries the involutive structure of the category of finite sets and relations into the involutive structure of the category of finite dimensional Hilbert spaces.

Enhancing the category **FinRel**.

By a vector bundle over a finite set we simply mean a rule which assigns a vector space E_x (which we will assume to be finite dimensional) to each point x of X. We are going to consider a category whose objects are vector bundles over finite sets. We will denote such an object by $E \to X$.

Following Atiyah and Bott, we will define the morphisms in this category as follows: If $E \to X$ and $F \to Y$ are objects in our category, and $\Gamma \subset X \times Y$ we consider the vector bundle over Γ which assigns to each point $(x, y) \in \Gamma$ the vector space $\mathrm{Hom}(F_y, E_x)$. A morphism in our category will be a section of this vector bundle.

So a morphism in our category will be a subset Γ of $X \times Y$ together with a map

$$r_{x,y} : F_y \to F_x$$

given for each $(x, y) \in \Gamma$. Suppose that $(\Gamma_1, r) \in \mathrm{Morph}(E \to X, F \to Y)$ and $(\Gamma_2, s) \in \mathrm{Morph}(F \to Y, G \to Z)$. Their composition is defined to be $(\Gamma_2 \circ \Gamma_1, t)$ where t is the section of the vector bundle over $\Gamma_2 \circ \Gamma_1$ given by

$$t(x, z) = \sum_{y|(x,y)\in\Gamma_1,(y,z)\in\Gamma_2} r(x, y) \circ s(y, z).$$

The verification of the category axioms is immediate.

We have **enhanced** the category of finite sets and relations to the category of vector bundles over finite sets.

Symplectic vector spaces.

Let V be a (usually finite dimensional) vector space over the real numbers. A symplectic structure on V consists of an antisymmetric bilinear form

$$\omega : V \times V \to \mathbf{R}$$

which is non-degenerate. So we can think of ω as an element of $\wedge^2 V^*$ when V is finite dimensional, as we shall assume until further notice. A vector space equipped with a symplectic structure is called a symplectic vector space.

A basic example:

A basic example is $\mathbf{R}^2$ with

$$\omega_{\mathbf{R}^2}\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) := \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

We will call this the standard symplectic structure on $\mathbf{R}^2$.

Special kinds of subspaces.

If W is a subspace of symplectic vector space V then $W^\perp$ denotes the symplectic orthocomplement of W:

$$W^\perp := \{v \in V |\ \omega(v, w) = 0,\ \forall w \in W\}.$$

A subspace is called

1. **symplectic** if $W \cap W^\perp = \{0\}$,
2. **isotropic** if $W \subset W^\perp$,
3. **coisotropic** if $W^\perp \subset W$, and
4. **Lagrangian** if $W = W^\perp$.

Linear canonical relations.

Let V_1 and V_2 be symplectic vector spaces with symplectic forms ω_1 and ω_2. We will let V_1^- denote the vector space V_1 equipped with the symplectic form $-\omega_1$. So $V_1^- \oplus V_2$ denotes the vector space $V_1 \oplus V_2$ equipped with the symplectic form $-\omega_1 \oplus \omega_2$.

A Lagrangian subspace Γ of $V_1^- \oplus V_2$ is called a **linear canonical relation**. The purpose of this section is to show that if we take the collection of symplectic vector spaces as objects, and the linear canonical relations as morphisms we get a category. Here composition is in the sense of composition of relations as in the category **FinRel**.

Let V_3 be a third symplectic vector space, let

$$\Gamma_1 \text{ be a Lagrangian subspace of } \quad V_1^- \oplus V_2$$

and let

$$\Gamma_2 \text{ be a Lagrangian subspace of } \quad V_2^- \oplus V_3.$$

Recall that as a *set* (see (3.1)) the composition

$$\Gamma_2 \circ \Gamma_1 \subset V_1 \times V_3$$

is defined by

$$(x, z) \in \Gamma_2 \circ \Gamma_1 \ \Leftrightarrow \ \exists\, y \in V_2 \text{ such that } (z, y) \in \Gamma_1 \text{ and}$$

$$(y, z) \in \Gamma_2.$$

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$$(y, z) \in \Gamma_2.$$

We must show that this is a Lagrangian subspace of $V_1^- \oplus V_3$. It will be important for us to break up the definition of $\Gamma_2 \circ \Gamma_1$ into two steps:

The space $\Gamma_2 \star \Gamma_1$.

Define

$$\Gamma_2 \star \Gamma_1 \subset \Gamma_1 \times \Gamma_2$$

to consist of all pairs $((x,y),(y',z))$ such that $y = y'$. We will restate this definition in two convenient ways. Let

$$\pi : \Gamma_1 \to V_2, \qquad \pi(v_1, v_2) = v_2$$

and

$$\rho : \Gamma_2 \to V_2, \qquad \rho(v_2, v_3) = v_2.$$

Let

$$\tau : \; \Gamma_1 \times \Gamma_2 \to V_2$$

$$\tau(\gamma_1, \gamma_2) := \pi(\gamma_1) - \rho(\gamma_2).$$

$$\pi : \Gamma_1 \to V_2, \qquad \pi(v_1, v_2) = v_2$$

$$\rho : \Gamma_2 \to V_2, \qquad \rho(v_2, v_3) = v_2.$$

$$\tau : \ \Gamma_1 \times \Gamma_2 \to V_2 \qquad \tau(\gamma_1, \gamma_2) := \pi(\gamma_1) - \rho(\gamma_2).$$

Then $\Gamma_2 \star \Gamma_1$ is determined by the exact sequence

$$0 \to \Gamma_2 \star \Gamma_1 \to \Gamma_1 \times \Gamma_2 \xrightarrow{\tau} V_2 \to \text{Coker}\, \tau \to 0. \quad (3.6)$$

Another way of saying the same thing is to use the language of “fiber products” or “exact squares”. Let $f : A \to C$ and $g : B \to C$ be maps, say between sets. Then we express the fact that $F \subset A \times B$ consists of those pairs (a, b) such that $f(a) = g(b)$ by saying that

$$\begin{array}{ccc} F & \longrightarrow & A \\ \downarrow & & \downarrow f \\ B & \underset{g}{\longrightarrow} & C \end{array}$$

is an **exact square** or a **fiber product diagram**.

Thus another way of expressing the definition of $\Gamma_2 \star \Gamma_1$ is to say that

$$\begin{array}{ccc} \Gamma_2 \star \Gamma_1 & \longrightarrow & \Gamma_1 \\ \downarrow & & \downarrow \pi \\ \Gamma_1 & \underset{\rho}{\longrightarrow} & V_2 \end{array} \quad . \tag{3.7}$$

is an exact square.

The projection $\alpha : \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_1$.

Consider the map

$$\alpha : (x, y, y, z). \mapsto (x, z). \tag{3.8}$$

By definition

$$\alpha : \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_1.$$

The kernel and image of a linear canonical relation.

Let V_1 and V_2 be symplectic vector spaces and let $\Gamma \subset V_1^- \times V_2$ be a linear canonical relation. Let

$$\pi : \Gamma \to V_2$$

be the projection onto the second factor. Define

- $\operatorname{Ker}\Gamma \subset V_1$ by $\operatorname{Ker}\Gamma = \{v \in V_1 | (v, 0) \in \Gamma\}$.
- $\operatorname{Im}\,\Gamma \subset V_2 = \Gamma(V_1)$.

So $\Gamma^\dagger \subset V_2^- \oplus V_1$ and hence both $\ker\Gamma^\dagger$ and $\operatorname{Im}\,\Gamma$ are linear subspaces of the symplectic vector space V_2. We claim that

$$(\ker\Gamma^\dagger)^\perp = \operatorname{Im}\,\Gamma. \tag{3.9}$$

- $\operatorname{Ker}\Gamma \subset V_1$ by $\operatorname{Ker}\Gamma = \{v \in V_1 | (v, 0) \in \Gamma\}$.
- $\operatorname{Im}\,\Gamma \subset V_2 = \Gamma(V_1)$.

$$(\ker \Gamma^{\dagger})^{\perp} = \operatorname{Im}\,\Gamma. \tag{3.9}$$

Here $\perp$ means perpendicular relative to the symplectic structure on V_2.

Proof. Let ω_1 and ω_2 be the symplectic bilinear forms on V_1 and V_2 so that $\tilde{\omega} = -\omega_1 \oplus (\omega_2)$ is the symplectic form on $V_1^- \oplus V_2$. So $v \in V_2$ is in $\operatorname{Ker}\Gamma^{\dagger}$ if and only if $(0, v) \in \Gamma$. Since Γ is Lagrangian,

$$(0, v) \in \Gamma^{\perp} \Leftrightarrow 0 = -\omega_1(0, v_1) + \omega_2(v, v_2) = \omega_2(v, v_2)\ \forall\ (v_1, v_2) \in \Gamma.$$

But this is precisely the condition that $v \in (\operatorname{Im}\,\Gamma)^{\perp}$.
□

The kernel of α .

The kernel of α consists of those $(0, v, v, 0) \in \Gamma_2 \star \Gamma_1$. We may thus identify

$$\ker \alpha = \ker \Gamma_1^\dagger \cap \ker \Gamma_2 \qquad (3.10)$$

as a subspace of V_2.

$$\tau : \ \Gamma_1 \times \Gamma_2 \to V_2 \qquad\qquad \tau(\gamma_1, \gamma_2) := \pi(\gamma_1) - \rho(\gamma_2).$$

$$\alpha : (x, y, y, z) \mapsto (x, z). \qquad\qquad \alpha : \Gamma_2 \star \Gamma_1 \to \Gamma_2 \circ \Gamma_1.$$

$$\ker \alpha = \ker \Gamma_1^\dagger \cap \ker \Gamma_2 \qquad (3.10)$$

$$\operatorname{Im} \tau = \operatorname{Im} \Gamma_1 + \operatorname{Im} \Gamma_2^\dagger, \qquad (3.11)$$

a subspace of V_2. If we compare (3.10) with (3.11) we see that

$$\ker \alpha = (\operatorname{Im} \tau)^\perp \qquad (3.12)$$

as subspaces of V_2 where $\perp$ denotes orthocomplement relative to the symplectic form ω_2 of V_2.

Proof that $\Gamma_2 \circ \Gamma_1$ is Lagrangian.

Since $\Gamma_2 \circ \Gamma_1 = \alpha(\Gamma_2 \star \Gamma_1)$ and $\Gamma_2 \star \Gamma_1 = \ker \tau$ it follows that $\Gamma_2 \circ \Gamma_1$ is a linear subspace of $V_1^- \oplus V_3$.

It is equally easy to see that $\Gamma_2 \circ \Gamma_1$ is an isotropic subspace of $V_1^- \oplus V_2$. Indeed, if (x, z) and (x', z') are elements of $\Gamma_2 \circ \Gamma_1$, then there are elements y and y' of V_2 such that

$$(x, y) \in \Gamma_1,\ (y, z) \in \Gamma_2,\ (x', y') \in \Gamma_1,\ (y', z') \in \Gamma_2.$$

Then

$$\omega_3(z, z') - \omega_1(x, x') =$$

$$\omega_3(z, z') - \omega_2(y, y') + \omega_2(y, y') - \omega_1(x, x') = 0.$$

$$\ker \alpha = (\operatorname{Im} \tau)^{\perp} \tag{3.12}$$

So we must show that $\dim \Gamma_2 \circ \Gamma_1 = \frac{1}{2} \dim V_1 + \frac{1}{2} \dim V_3$. It follows from (3.12) that

$$\dim\ \ker\ \alpha = \dim\ V_2 - \dim \operatorname{Im}\ \tau$$

and from the fact that $\Gamma_2 \circ \Gamma_1 = \alpha(\Gamma_2 \star \Gamma_1)$ that

$$\dim\ \Gamma_2 \circ \Gamma_1 = \dim\ \Gamma_2 \star \Gamma_1 - \dim\ \ker \alpha =$$

$$= \dim\ \Gamma_2 \star \Gamma_1 - \dim\ V_2 + \dim \operatorname{Im}\ \tau.$$

$$\dim\ \Gamma_2 \circ \Gamma_1 = \dim\ \Gamma_2 \star \Gamma_1 - \dim\ \ker \alpha =$$

$$= \dim\ \Gamma_2 \star \Gamma_1 - \dim\ V_2 + \dim \operatorname{Im}\ \tau.$$

Since $\Gamma_2 \star \Gamma_1$ is the kernel of the map $\tau : \Gamma_1 \times \Gamma_2 \to V_2$ if follows that

$$\dim\ \Gamma_2 \star \Gamma_1 = \dim \Gamma_1 \times \Gamma_2 - \dim\ \operatorname{Im}\ \tau =$$

$$\frac{1}{2} \dim\ V_1 + \frac{1}{2} \dim\ V_2 + \frac{1}{2} \dim\ V_2 + \frac{1}{2} \dim\ V_3 - \dim\ \operatorname{Im}\ \tau.$$

Putting these two equations together we see that

$$\dim \Gamma_2 \circ \Gamma_1 = \frac{1}{2} \dim V_1 + \frac{1}{2} \dim V_3$$

as desired.

Main theorem of today:

Theorem 6 *The composite* $\Gamma_2 \circ \Gamma_1$ *of two linear canonical relations is a linear canonical relation.*

The diagonal Δ_V gives the identity morphism and so we have verified that **LinSym** is a category whose objects are symplectic vector spaces and whose morphisms are linear canonical relations.

The category **LinSym** and the symplectic group.

The category **LinSym** is a vast generalization of the symplectic group because of the following observation: Let X and Y be symplectic vector spaces. Suppose that the Lagrangian subspace $\Gamma \subset X^- \oplus Y$ projects bijectively onto X under the projection of $X \oplus Y$ onto the first factor. This means that Γ is the graph of a linear transformation T from X to Y:

$$\Gamma = \{(x, Tx)\}.$$

T must be injective. Indeed, if $Tx = 0$ the fact that Γ is isotropic implies that $x \perp X$ so $x = 0$. Also T is surjective since if $y \perp \mathrm{im}(T)$, then $(0, y) \perp \Gamma$. This implies that $(0, y) \in \Gamma$ since Γ is maximal isotropic.

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$$\omega_Y(Tx_1, Tx_2) = \omega_X(x_1, x_2),$$

i.e. T is a symplectic isomorphism. If $\Gamma_1 = \operatorname{graph} T$ and $\Gamma_2 = \operatorname{graph} S$ then

$$\Gamma_2 \circ \Gamma_1 = \operatorname{graph} S \circ T$$

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$$\Gamma_2 \circ \Gamma_1 = \operatorname{graph} S \circ T$$

so composition of Lagrangian relations reduces to composition of symplectic isomorphisms in the case of graphs. In particular, if we take $Y = X$ we see that $\mathrm{Symp}(X)$ is a subgroup of $\mathrm{Morph}\,(X, X)$ in our category.