Symplectic Geometry Lecture 2

Properties of the symplectic group

Review.

Let V be a (usually finite dimensional) vector space over the real numbers. A symplectic structure on V consists of an antisymmetric bilinear form

$$\omega : V \times V \to \mathbf{R}$$

which is non-degenerate. So we can think of ω as an element of $\wedge^2 V^*$ when V is finite dimensional, as we shall assume until further notice. A vector space equipped with a symplectic structure is called a symplectic vector space.

A basic example is $\mathbf{R}^2$ with

$$\omega_{\mathbf{R}^2}\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) := \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

We will call this the standard symplectic structure on $\mathbf{R}^2$.

Special kinds of subspaces.

If W is a subspace of symplectic vector space V then $W^\perp$ denotes the symplectic orthocomplement of W:

$$W^\perp := \{v \in V|\ \omega(v, w) = 0,\ \forall w \in W\}.$$

A subspace is called

1. **symplectic** if $W \cap W^\perp = \{0\}$,
2. **isotropic** if $W \subset W^\perp$,
3. **coisotropic** if $W^\perp \subset W$, and
4. **Lagrangian** if $W = W^\perp$.

Review: normal forms.

For any non-zero $e \in V$ we can find an $f \in V$ such that $\omega(e, f) = 1$ and so the subspace W spanned by e and f is a two dimensional symplectic subspace. Furthermore the map

$$e \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

gives a symplectic isomorphism of W with $\mathbf{R}^2$ with its standard symplectic structure. We can apply this same construction to $W^\perp$ if $W^\perp \neq 0$. Hence by induction, we can decompose any symplectic vector space into a direct sum of two dimensional symplectic subspaces:

$$V = W_1 \oplus \cdots W_d$$

All symplectic vector spaces of the same dimension are isomorphic.

$$V = W_1 \oplus \cdots W_d$$

where $\dim V = 2d$ (proving that every symplectic vector space is even dimensional) and where the W_i are pairwise (symplectically) orthogonal and where each W_i is spanned by e_i, f_i with $\omega(e_i, f_i) = 1$. In particular this shows that all $2d$ dimensional symplectic vector spaces are isomorphic, and isomorphic to a direct sum of d copies of $\mathbf{R}^2$ with its standard symplectic structure.

Existence of Lagrangian subspaces.

Let us collect the $e_1, \ldots, e_d$ in the above construction and let L be the subspace they span. It is clearly isotropic. Also, $e_1, \ldots, e_n, f_1, \ldots, f_d$ form a basis of V. If $v \in V$ has the expansion

$$v = a_1 e_1 + \cdots a_d e_d + b_1 f_1 + \cdots + b_d f_d$$

in terms of this basis, then $\omega(e_i, v) = b_i$. So $v \in L^\perp \Rightarrow v \in L$. Thus L is Lagrangian. So is the subspace M spanned by the f's.

Conversely, if L is a Lagrangian subspace of V and if M is a complementary Lagrangian subspace, then ω induces a non-degenerate linear pairing of L with M and hence any basis $e_1, \cdots e_d$ picks out a dual basis $f_1, \cdots .f_d$ of M giving a basis of V of the above form.

Consistent Hermitian structures.

In terms of the basis $e_1, \ldots, e_n, f_1, \ldots, f_d$ introduced above, consider the linear map

$$J: \quad e_i \mapsto -f_i, \quad f_i \mapsto e_i.$$

It satisfies

$$\begin{aligned} J^2 &= -I, & (1) \\ \omega(Ju, Jv) &= \omega(u, v), \quad \text{and} & (2) \\ \omega(Ju, v) &= \omega(Jv, u). & (3) \end{aligned}$$

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\end{aligned}
$$

Notice that any J which satisfies two of the three conditions above automatically satisfies the third. Condition (1) says that J makes V into a d-dimensional complex vector space. Condition (2) says that J is a symplectic transformation, i.e acts so as to preserve the symplectic form ω. Condition (3) says that $\omega(Ju, v)$ is a real symmetric bilinear form.

All three conditions (really any two out of the three) say that $(\ , \ ) = (\ , \ )_{\omega,J}$ defined by

$$(u, v) = \omega(Ju, v) + i\omega(u, v)$$

is a semi-Hermitian form whose imaginary part is ω. For the J chosen above this form is actually Hermitian, that is the real part of $(\ , \ )$ is positive definite.

Compatible complex structures.

For the proof of various facts about the symplectic group, it will be convenient to focus attention on the positive definite case.

Let V be a symplectic vector space with symplectic form ω.

Recall that a complex structure on a vector space V is an automorphism $J : V \to V$ such that $J^2 = -\,\mathrm{Id}$.

Definition 1 *A complex structure J on a symplectic vector space V is called (ω) compatible if*

$$g(v, w) := \omega(Jv, w)$$

defines a positive definite inner product on V.

A compatible complex structure makes V into a Hermitian vector space that is, into a complex inner product space with Hermitian metric

$$h(v, w) = g(v, w) + i\omega(v, w).$$

Starting with such a compatible complex structure, let $e_1, \ldots, e_n$ be an orthonormal basis of V thought of as an n-dimensional complex vector space with Hermitian form h. Let $f_j := -Je_j$. Then $\omega(e_j, e_k) = \omega(f_j, f_k) = 0$ for all j, k and

$$\omega(e_i, f_j) = \operatorname{Im} h(e_i, -Je_j) = \delta_{ij}.$$

So the e, f form a symplectic basis of V when we think of V as a real symplectic vector space.

Review: polar decomposition.

We recall the following facts from linear algebra. Let V be a real vector space with a positive definite scalar product, g. If $A : V \to V$ is linear transformation its adjoint is defined by $g(Au, v) = g(u, A^*v)$ for all $u, v \in V$. If $A = A^*$ we say that A is self-adjoint, and we say that a linear transformation O is orthogonal if $OO^* =$Id.

It is a theorem that every self-adjoint matrix C can be diagonalized - more precisely, that there is a decomposition of V into a direct sum of mutually orthogonal subspaces such that the restriction of C to each subspace is multiplication by a real number. If $C = B^2$ where B is self-adjoint (so that C is also), then the decomposition for B works for C. In particular, if C is non-negative ($g(Cu, u) \geq 0$ for all u) then C has a unique non-negative square root.

Proposition 1 *Let* $A : V \to V$ *be an invertible linear transformation on a real vector space with a positive definite scalar product,* g*. Then we can write*

$$A = PO$$

where P *is positive definite and* O *is orthogonal, and this decomposition is unique.*

Proof. The operator AA^* is self-adjoint and positive. So it has a unique positive square root P. If we had $A = PO$ then $AA^* = P^2$ showing that P and hence O is unique. If we take P to be the square root of AA^* then $(P^{-1}A)\cdot(P^{-1}A)^* = P^{-1}AA^*P^{-1}$ showing that $O := P^{-1}A$ is orthogonal. $\square$

Using the polar decomposition.

For any real vector space V let $\mathrm{Riem}(V)$ denote the convex open subset of the space $S^2(V^*)$ consisting of all positive definite symmetric bilinear forms on V.

Now let (V, ω) be a symplectic vector space. We have associated to each compatible complex structure on V an element g of $\mathrm{Riem}(V)$. So we have defined a map

$$G : \mathcal{J}(V, \omega) \to \mathrm{Riem}(V), \quad J \mapsto g.$$

On the other hand, for every $k \in \mathrm{Riem}(V)$, there is a linear transformation $A : V \to V$ which is uniquely defined by

$$k(u, v) = \omega(Au, v).$$

Since ω is anti-symmetric it follows that A is skew-adjoint ($A = -A^*$) with respect to k. Since ω is non-singular, A is invertible.

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Since ω is anti-symmetric it follows that A is skew-adjoint ($A = -A^*$) with respect to k. Since ω is non-singular, A is invertible. Let us apply the polar decomposition

$$A = PO$$

to A. We have $A = -A^* = -O^*P = (O^*PO)(-O^{-1})$. By uniqueness, of the polar decomposition. We conclude that $OP = PO$ and $O = -O^{-1}$. This second condition says that $O^2 = -\operatorname{Id}$ so O is a complex structure which we shall now denote by J. So $A = PJ = JP$ and $J^2 = -\operatorname{Id}$.

We claim that J is ω-compatible. Indeed

$$\omega(Ju, v) = \omega(AP^{-1}u, v) = k(P^{-1}u, v) = k(u, P^{-1}v) = k(P^{-\frac{1}{2}}u, P^{-\frac{1}{2}}v).$$

Thus $g(u, v) := k(P^{-\frac{1}{2}}u, P^{-\frac{1}{2}}v)$ is positive definite, and hence J is ω-compatibile. So we have proved:

Theorem 1 *The map $k \mapsto J$ defined above is a map*

$$F : \mathrm{Riem}(V) \to \mathcal{J}(V, \omega)$$

and

$$F \circ G = \mathrm{id}\,.$$

The space $\mathcal{J}(V, \omega)$ has a topology (inherited from the topology of the space of all linear transformations of V) as does the space $\mathrm{Riem}(V)$ (as an open convex set in $S^2(V^*)$). The maps G and F are clearly continuous. Since $\mathrm{Riem}(V)$ is convex, and hence contractible we conclude that $\mathcal{J}(V, \omega)$ is contractible.

Suppose we fix $J \in \mathcal{J}(V, \omega)$ and a Lagrangian subspace L of V. We may choose a basis $e_1, \ldots, e_n$ of L which is orthonormal relative to the metric $g = G(J)$. Then $e_1, \ldots, e_n$ is an orthonormal basis of V relative to the associated Hermitian form (and complex structure). If L' is another Lagrangian subspace and $e'_1, \ldots, e'_n$ an orthonormal basis of it, then there will be a unitary map U such that $Ue_j = e'_j$ for all j. So the group $U(V, J, h)$ which is a subgroup of $Sp(V)$ acts transitively on the space of Lagrangian subspaces. The stabilizer group of L consists of those unitary transformations which are real. So if $\dim V = 2n$ we can say that the space of all Lagrangian subspaces of V is diffeomorphic to $U(n)/O(n)$. In particular, it is a manifold of dimension

$$n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

The group $Sp(V)$ is connected.

Fix a compatible complex structure J so that the corresponding unitary group (which we may denote by $U(V,J)$) is the subgroup preserving J. If J' is another compatible complex structure, the (real) linear transformation which carries an orthonormal basis for the Hermitian structure of J into one for J' is symplectic. Hence $Sp(V)$ acts transitively on $\mathcal{J}(V,\omega)$ and

$$\mathcal{J}(V,\omega) = Sp(V)/U(V,J).$$

Since $\mathcal{J}(V,\omega)$ is contractible and $U(n)$ is connected we see that $Sp(V)$ is connected.

The dimension of $Sp(V)$.

All symplectic structures on V are equivalent. This means that $Gl(V)$ acts transitively on the open subset of $\wedge^2(V^*)$ consisting of symplectic structures. The dimension of $\wedge^2(V^*)$ is $\frac{2n(2n-1)}{2}$ and the dimension of $GL(v)$ is $(2n)^2$. So the dimension of $Sp(V)$ is $\frac{2n(2n+1)}{2}$.

A coordinate description of $Sp(V)$.

Suppose we fix a compatible complex structure whose associated symmetric form is g. We let $M^\dagger$ denote the transpose of $M \in Gl(V)$ with respect to g.

Now $M \in Sp(V)$ if and only if $\omega(Mu, Mv) = \omega(u, v) \;\; \forall\, u, v \in V$. Since $\omega(u, v) = g(u, Jv)$ this says that $g(Mu, JMv) = g(u, Jv)$ or

$$M^\dagger JM = J.$$

Another way of writing this is

$$M^\dagger = JM^{-1}J^{-1}.$$

In terms of a standard symplectic basis our choice of J will have the block decompositon

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

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where I is the $n \times n$ identity matrix. Suppose we use the corresponding block decomposition for M:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

so that

$$M^\dagger = \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix}$$

where A^t denotes the transpose of A as an $n \times n$ matrix etc. So

$$\begin{aligned} M^\dagger J M &= \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= \begin{pmatrix} C^t A - A^t C & C^t B - A^t D \\ D^t A - B^t C & D^t B - B^t D \end{pmatrix}. \end{aligned}$$

$$M^\dagger JM = \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$
$$= \begin{pmatrix} C^tA - A^tC & C^tB - A^tD \\ D^tA - B^tC & D^tB - B^tD \end{pmatrix}.$$

$$M^\dagger JM = J.$$

So the condition for M to be symplectic is

$$A^tC = C^tA, \quad B^tD = D^tB, \text{ and } A^tD - C^tB = I.$$

Since $\det J = 1$ the condition $M^\dagger JM = J$ implies that $(\det M)^2 = 1$. Since $Sp(V)$ is connected, this implies that

$$\det M = 1.$$

Eigenvalues of a symplectic matrix.

For any real matrix, its complex eigenvalues occur in complex conjugate pairs. The eigenvalues of $M^\dagger$ are the same as the eigenvalues of M. But $M^\dagger = JM^{-1}J^{-1}$ so $M^\dagger$ and M^{-1} have the same eigenvalues. So if λ is an eigenvalue of M so is λ^{-1}. So

Theorem 2 *The eigenvalues of $M \in Sp(V)$ occur as either*

- *real pairs λ and λ^{-1},* $\lambda \neq \pm 1$, *or*
- *complex pairs λ and $\lambda^{-1} = \overline{\lambda}$,* $\lambda \neq \pm 1$, *or*
- *complex quadruples: λ not real, $|\lambda| \neq 1$,*

$$\lambda,\ \lambda^{-1},\ \overline{\lambda},\ \overline{\lambda}^{-1}, \quad or,$$

- $\lambda = -1$ *or*
- $\lambda = 1$.

The multiplicity of -1 and of 1 is even.

Proof of the theorem.

We have already verified that the eigenvalues which are not $= \pm 1$ are of the first three types. The product of all the eigenvalues of these types is 1, and $\det M = 1$, so the multiplicity of -1 is even. There are an even number of eigenvalues of the first four types and the size of M is even, so the multiplicity of 1 must also be even.

The Lie algebra of $Sp(V)$.

By definition, this consists of all linear transformations ξ of V such that the corresponding one parameter group

$$\exp t\xi := I + t\xi + \frac{1}{2}t^2\xi^2 + \frac{1}{3!}t^3\xi^3 + \cdots$$

is a subgroup of $Sp(V)$. Differentiating the equation

$$\omega(\exp t\xi u, \exp t\xi v) = \omega(u, v)$$

with respect to t and setting $t = 0$ gives the condition

$$\omega(\xi u, v) + \omega(u, \xi v) = 0, \quad \forall \;\; u, v \in V$$

as the condition for ξ to belong to the Lie algebra of $Sp(V)$. We denote this Lie algebra by $sp(V)$. (The above condition is also sufficient as can be seen by solving a linear differential equation.)

Polar decomposition of elements of $Sp(V)$.

Fix a compatible complex structure and hence a corresponding positive definite scalar product g on V. Every invertible linear transformation the has a polar decomposition relative to g. Let $M \in Sp(V)$ and

$$M = PO$$

its polar decomposition. I wish to show that P and O both belong to $Sp(V)$. Then since O preserves both ω and g it belongs to $U(V, J)$. As J is fixed I will denote $U(V, J)$ by $U(V)$. I will also denote the set of positive definite matrices belonging to $Sp(V)$ by $\mathbf{P}$. We will show that the map

$$\mathbf{P} \times U(V) \to Sp(V), \quad (P, O) \mapsto PO$$

is a diffeomorphism and that the space $\mathbf{P}$ is contractible. This implies that $Sp(V)$ is homotopically equivalent to $U(V)$. Since the fundamental group of $U(V)$ is $\mathbb{Z}$ we conclude that the fundamental group of $Sp(V)$ is also $\mathbb{Z}$.

Suppose we start with $M \in Sp(V)$. We know that $M^\dagger = JM^{-1}J^{-1} \in Sp(V)$ and hence that $MM^\dagger \in \mathbf{P}$. Since the P in the polar decomposition of M is given by taking the positive definite square root $MM^\dagger$, we must show that the positive definite square root of an element of $\mathbf{P}$ belongs to $\mathbf{P}$.

For this we begin with a lemma:

Lemma 1 *Let $M \in Sp(V)$ be symmetric, i.e. $M = M^\dagger$ so that all the eigenvalues λ of M are real. Let V_λ denote the eigenspace corresponding to λ. Then*

$$V_\lambda^\perp = \bigoplus_{\mu | \mu\lambda \neq 1} V_\mu .$$

To prove:

$$V_\lambda^\perp = \bigoplus_{\mu|\mu\lambda\neq 1} V_\mu.$$

Proof. For $u \in V_\lambda$ and $v \in V_\mu$ we have

$$\omega(u,v) = \omega(Mu, Mv) = \lambda\mu\omega(u,v).$$

So if $\mu\lambda \neq 1$, then $V_\mu \subset V_\lambda^\perp$. We now consider the various possibilities for λ. We will find in all cases that the sum of the dimensions of the V_μ, $\mu\lambda \neq 1$ is $\dim V - \dim V_\lambda$ which will prove the lemma, since $\dim V_\lambda^\perp = \dim V - \dim V_\lambda$

If $\lambda = 1$, then this sum of dimensions is the sum of the dimensions of the eigenspaces corresponding to eigenvalues $\neq 1$. So we are ok. Notice that in this case V_1 is a symplectic subspace of V. Similarly for $\lambda = -1$. If $\lambda \neq \pm 1$ then the μ in question consist of all $\mu \neq \lambda^{-1}$. But, by the theorem, the multiplicity of λ^{-1} is the same as the multiplicity of λ, so $\dim V - \dim V_\lambda = \dim V - \dim V_{\lambda^{-1}}$ and since M is diagonalizable, this is the sum of the dimensions of V_μ, $\mu \neq \lambda^{-1}$. So we are also ok in this case. Notice that in the case $\lambda \neq \pm 1$, the space V_λ is isotropic but $V_\lambda \oplus V_{\lambda^{-1}}$ is symplectic. $\square$

Now suppose that $T \in \mathbf{P}$ so that its eigenvalues are positive. T acts as the identity on the space V_1, so that on this subspace $T = \exp 0$ and so the entire one parameter group $T^s = I$ on this subspace.

On the subspace $V_\lambda \oplus V_{\lambda^{-1}}$ the operator T has the block decomposition

$$T = \begin{pmatrix} \lambda I & 0 \\ 0 & \lambda^{-1} I \end{pmatrix}.$$

So if $\sigma = \log \lambda$ the entire one parameter group

$$T^s := \begin{pmatrix} e^{s\sigma} I & 0 \\ 0 & e^{-s\sigma} I \end{pmatrix}$$

belongs to $Sp(V_\lambda \oplus V_{\lambda^{-1}})$.

In particular, every $T \in \mathbf{P}$ lies on a one parameter group of elements all lying in $\mathbf{P}$. Indeed, if we let $\mathfrak{p} \subset sp(V)$ denote the set of symmetric elements, i.e. those satisfying $\xi^\dagger = \xi$, we have shown that the exponential map exp:

$$\xi \mapsto \exp \xi$$

restricts to a diffeomorphism

$$\exp : \mathfrak{P} \to \mathbf{P}.$$

Let us now go back the the polar decomposition of an element M of $Sp(V)$. We now know that $P = (MM^\dagger)^{\frac{1}{2}}$ belongs to $\mathbf{P}$ and since $M = PO$ that $O \in Sp(V)$ so that in fact $O \in U(V)$. We have proved all our claims.

The Cartan decomposition of $sp(V)$

Let $\mathfrak{g} := sp(V)$ and $\mathfrak{k} := u(V)$, the Lie algebra of $U(V)$. From the above polar decomposition we see that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

as vector spaces. Since $U(V) = Sp(V) \cap O(V)$ it is clear that $\mathfrak{p}$ is stable under $U(V)$ under conjugation and hence that

$$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}.$$

Since the commutator of two symmetric matrices is antisymmetric, we have

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Also $[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}$ since $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$. So we have

$$\begin{aligned} \mathfrak{g} &= \mathfrak{k} \oplus \mathfrak{p} \\ [\mathfrak{k},\mathfrak{k}] &\subset \mathfrak{k} \\ [\mathfrak{k},\mathfrak{p}] &\subset \mathfrak{p} \\ [\mathfrak{p},\mathfrak{p}] &\subset \mathfrak{k}. \end{aligned}$$

This is an example of what is known as a Cartan decomposition of a real semi-simple Lie algebra

A "Gauss" decomposition for symplectic matrices.

I wish to show that matrices of the form

$$\begin{pmatrix} I & dI \\ 0 & I \end{pmatrix}, \qquad d \in \mathbb{R}$$

and

$$\begin{pmatrix} I & 0 \\ S & I \end{pmatrix}, \qquad S = S^t$$

generate the symplectic group. This will take some computation. But a consequence of this fact will be the group $Sp(4)$ corresponds to linear optics just as $Sp(2)$ corresponds to Gaussian optics. Indeed, the (4×4) matrix $\begin{pmatrix} I & dI \\ 0 & I \end{pmatrix}$ corresponds to straight line propagation in a medium of constant index of refraction while the matrix $\begin{pmatrix} I & 0 \\ S & I \end{pmatrix}$ corresponds to a linearized version of Snell's law at a (parabolic) surface.

$$\begin{pmatrix} I & dI \\ 0 & I \end{pmatrix}, \qquad d \in \mathbb{R}$$

and

$$\begin{pmatrix} I & 0 \\ S & I \end{pmatrix}, \qquad S = S^t$$

Let G be the subgroup generated by these matrices. We wish to show that $G = Sp(V)$.

First

$$\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

so

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in G.$$

Next

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

so

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \in G.$$

Next

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}$$

so all matrices of the form

$$\begin{pmatrix} I & S \\ 0 & I \end{pmatrix}, \quad S = S^t$$

belong to G.

If $S = S^t$ and is invertible, we have

$$\begin{pmatrix} I & S^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -S & I \end{pmatrix} \begin{pmatrix} I & S^{-1} \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & S^{-1} \\ -S & 0 \end{pmatrix}$$

so all matrices of the form

$$\begin{pmatrix} 0 & S^{-1} \\ -S & 0 \end{pmatrix}$$

with S symmetric and invertible belong to G.

Now

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} 0 & S^{-1} \\ -S & 0 \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix}$$

so all matrices of the form

$$\begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix}$$

with S symmetric and invertible belong to G.

We claim the following lemma:

Lemma 2 *Every non-singular $n \times n$ matrix can be written as the product of three non-singular symmetric matrices.*

Lemma 2 *Every non-singular $n \times n$ matrix can be written as the product of three non-singular symmetric matrices.*

Assuming the lemma, we find that any matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$$

belongs to G. Indeed, write $A = S_1 S_2 S_3$ with the $S_i = S_i^t$. Then $(A^t)^{-1} = S_1^{-1} S_2^{-1} S_3^{-1}$ and each of the matrices

$$\begin{pmatrix} S_i & 0 \\ 0 & S_i^{-1} \end{pmatrix}$$

belongs to G.

Suppose that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(V)$$

with A non-singular. Then

$$\begin{pmatrix} I & 0 \\ -E & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C - EA & D - EB \end{pmatrix}.$$

If we choose $E = CA^{-1}$ we get $C - EA = 0$. Now $A^t C = C^t A$ which implies that CA^{-1} is symmetric. So for this choice of E we see that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ E & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - EB \end{pmatrix}.$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ E & I \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D - EB \end{pmatrix}.$$

We know that the first factor belongs to G. In the second factor we know that $D - EB = (A^t)^{-1}$ because the full matrix belongs to $Sp(V)$. Now

$$\begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}$$

and both factors on the right belong to G since $A^{-1}B$ is symmetric.

So every matrix in $Sp(V)$ with A non-singular belongs to G.

Now we consider the case where A is singular, say of rank $r < n$. Row and column reduction says that we can find non-singular $n \times n$ matrices Q and R such that

$$RAQ = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

By pre and post multiplying by

$$\begin{pmatrix} R & 0 \\ 0 & (R^t)^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q & 0 \\ 0 & (Q^t)^{-1} \end{pmatrix}$$

which belong to G we can arrange that we wish to show that matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

belong to G.

$$A = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

Write down the corresponding block decomposition for C so

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}.$$

$$A^t C = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix}$$

and the condition that $A^t C$ be symmetric implies that $C_2 = 0$ and that $C_1 = C_1^t$.

Now

$$\begin{pmatrix} I & E \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A + EC & B + ED \\ C & D \end{pmatrix}.$$

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Choose

$$E = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}.$$

Then E is symmetric and

$$A + EC = \begin{pmatrix} I_r & 0 \\ C_3 & C_4 \end{pmatrix}.$$

I claim that this matrix is non-singular, which would then complete the proof that $G = Sp(V)$ (up to the proof of the lemma). Indeed, if this matrix were singular, it would mean that C_4 is singular, which would mean that there is a non-zero vector v whose first r components vanish, and is sent into 0 by C (since $C_2 = 0$). But then $Av = 0$ and $Cv = 0$ contradicting the condition $D^t A - B^t C = I$.

Proof of the Lemma. To prove: any invertible $n \times n$ matrix A can be written as the product of three symmetric matrices: First use polar decomposition to write $A = PO$ where P is positive definite (in particular symmetric) and O is orthogonal. So we must show that every orthogonal matrix O can be written as the product of two symmetric matrices. We can block diagonalize O, that is write $O = RBR^{-1}$ where R is orthogonal and B consists of a block of matrices along the diagonal where each block is either a two by two block of an anti-symmetric matrix or is a one by one block(of ± 1). If $B = S_1 S_2$ then $RBR^{-1} = (RS_1 R^{-1})(RS_2 R^{-1})$ and the conjugate of a symmetric matrix by an orthogonal matrix is symmetric. So we are reduced to the block diagonal case, which means that we are reduced to the one or two dimensional case. A one by one matrix is always symmetric while

$$\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}. \qquad \square$$

2.4 Compact subgroups of $Sp(V)$.

Suppose that H is a compact subgroup of $Sp(V)$. By averaging over H we can find a $k \in \mathrm{Riem}(V)$ which is invariant under all elements of H, and hence so is the corresponding J. This means that H is a subgroup of the corresponding $U(V, J)$. Since $Sp(V)$ acts transitively on $\mathcal{J}(V, \omega)$, all such $U(V, J)$ are conjugate. So we have proved that every compact subgroup of $Sp(V)$ is a is a subgroup of $U(V, J)$ for some compatible J and all $U(V, J)$ are conjugate.