Symplectic geometry
Lecture 16

Collective motion

The results of the first part of this lecture are taken from:

Guillemin, V. and S. Sternberg [1980], "The moment map and collective motion", *Ann. of Phys*.,**127**, 220–253.

The rest of this lecture is taken from the paper "The centralizer of invariant functions and the division properties of moment maps" by Karshon and Lerman *Ill. J. Math.* **41**, no. 3 (1997), 462-487.

In this lecture I will attempt to give a mathematical formulation to the notion of a "model" of a given mechanical system. An example of what we have in mind is to try to explain what we mean when we say that a system of point particles moves under a given Hamiltonian, "as if if it were a rigid body." More generally, what do we mean when we say that one physical system moves "as if it were" some other physical system? Thus, for example, in nuclear physics one has the "liquid drop model" of the nucleus in which it is assumed that an appropriate "model" for the mechanical system which consists of N nucleons (considered as point particles) is a "liquid drop" - that the nucleus behaves "as if it were a liquid drop"'. Here, of course, we must explain what exactly we mean by the mechanical system consisting of a "liquid drop," but we also must explain, in more generality, what we mean by one system behaving as if it were another.

Example: the rigid body.

To get a grip on this problem, let us examine the intuitive and familiar case of the rigid body. Suppose we had a system of N point particles. We can imagine that they are (almost) rigidly attached to one another, if there is some potential energy $V = \sum V_{ij}$ where V_{ij} is a function of the distance between the ith and jth particles and takes on a very sharp minimum at certain specified distances r_{ij}. So we can imagine that the Hamiltonian of such a system has the form

$$V + K + K' + H$$

where V is the potential energy described above, K is an internal kinetic energy involving the motion of the points relative to one another, K' is some kinetic energy of the overall center of mass, and H is a Hamiltonian (involving both kinetic and potential terms) but is a function solely of the total angular momentum and inertia tensor relative to the center of mass.

The contribution of K' is simply to give an overall linear motion to the center of mass, and so by introducing coordinates relative to the center of mass we can ignore it. (This is an elementary example of reduction relative to the action of the translation group $\mathbb{R}^3$ acting simultaneously on all particles. Here Marsden-Weinstein reduction and orbital reduction coincide, since $\mathbb{R}^3$ is commutative and so its orbits are points.) Therefore, we are looking at a Hamiltonian of the form

$$V + K + H.$$

Then the actual motion of the system would be described, approximately, as a "rigid-body motion" coming from H, together with a superimposed rapid oscillation coming from $V + K$. To a good approximation, we might expect to be able to ignore the rapid oscillation, which averages itself out over the rigid-body motion.

$$V + K + H.$$

Then the actual motion of the system would be described, approximately, as a "rigid-body motion" coming from H, together with a superimposed rapid oscillation coming from $V + K$. To a good approximation, we might expect to be able to ignore the rapid oscillation, which averages itself out over the rigid-body motion. The situation is even more convincing in quantum mechanics. There, one might expect that for relatively low energies one can assume that the internal state of the system, and so the observed spectrum, should look like that of a rigid rotor. The fact that one sees unmistakable rotational levels in complicated nuclear spectra supports this description of the system. Thus the rigid-body description comes from the property that H is a function of the total angular momentum and inertia tensor,

the rigid-body description comes from the property that H is a function of the total angular momentum and inertia tensor, or quadrupole moment: Let $q_1, \ldots, q_N$ be the position vectors (relative to the center of mass) and let $p_1 \ldots, p_N$ be the corresponding momenta. We can then define the functions L and Q on the total phase space by

$$L := \sum p_i \wedge q_i, \qquad Q := \sum q_i \otimes q_i.$$

The assumption about H is that it is a function of L and Q (i.e. that $H = H(L, Q)$. Any such function can be said to define a "collective rigid-body Hamiltonian."

Now the important point I want to make is, as we saw in the last lecture, we can regard L and Q as components of a certain moment map, more precisely of a certain moment map of a semi-direct product subgroup of a symplectic group. This leads to the following definition:

The definition of a collective Hamiltonian.

Let (M, ω_M, Φ_M) be a Hamiltonian G-space.

Definition 1 *A function $\mathcal{H}$ on M of the form*

$$\mathcal{H} = F \circ \Phi_M$$

where F is a function on $\mathfrak{g}^*$ is called **collective**.

The liquid drop model.

For example, in the liquid drop model of the nucleus, one wants to imagine that the system of point particles behaves as an incompressible fluid, but that only the "quadratic approximation" to the shape of the liquid drop is what matters; in more mathematical language, what this means is that only the quadrupole moments matter. Thus, a possible configuration of a liquid drop is specified by a positive definite symmetric tensor $Q \in S^2(\mathbb{R}^3)$ and thegroup $Sl(3,R)$ acts on $S^2(\mathbb{R}^3)$ as in the last lecture. The fact that we use $Sl(3,\mathbb{R})$ and not $Gl(3)$ is the expression of the fact that we are dealing with a"liquid" drop, and not a "gaseous" drop; in other words, that "volume" is preserved. In fact, $\det(AQA^t = \det Q$ if $\det A = 1$,and so $\det Q$,thought of as the volume of the liquid drop is an invariant. If Q is positive definite, we can, by an appropriate A, bring Q to the form cI, where $c = (\det Q)^{\frac{1}{3}} > 0$.

As we saw, the isotropy group in $Sl(3,R)$ of cI is $SO(3)$, and so the corresponding orbit of Q in $S^2(\mathbb{R}^3)$ is five-dimensional. Such an orbit N is the configuration space of the liquid drop of given volume; notice that cI is the unique point in N that is left fixed by $SO(3)$ - it corresponds to a spherical globule of liquid.

Solving Hamilton's equations for a collective Hamiltonian.

Reminder: the Legendre transformation.

We now explain in general, what ingredients go into the solution of a "collective" Hamiltonian (according to Definition 1). For this purpose, we remind the reader that a smooth function F defined on an open subset U of a vector space (in this case the vector space is $\mathfrak{g}*$) defines a map $\mathcal{L}_F$, the Legendre transformation, of U into the dual space

$$\mathcal{L}_F : \ \mathfrak{g}^* \to (\mathfrak{g}^*)^* = \mathfrak{g}$$

by the formula

$$\langle \nu, \mathcal{L}_F(\mu) \rangle = \frac{d}{dt} F(\mu + t\nu)_{|t=0}. \qquad (1)$$

Two ways of constructing a vector field on M .

We now have two ways of constructing a vector field on M out of a function F on $\mathfrak{g}^*$:

We can form the collective Hamiltonian $\mathcal{H} = F \circ \Phi_M$ and then the Hamiltonian vector field $X_{\mathcal{H}}$. Or, we can proceed as follows:

At each $m \in M$ we apply the map Φ_M to get a point $\Phi_M(m) \in \mathfrak{g}^*$. We then apply the Legendre transformation $\mathcal{L}_F$ to get a point

$$\mathcal{L}_F\left(\Phi_M(m)\right) \in \mathfrak{g}$$

Each $A \in \mathfrak{g}$ determines a vector field A_M on M and hence a tangent vector $A_M(m) \in T_m M$. In particular we can do this for $A = \mathcal{L}_F\left(\Phi_M(m)\right)$. We can do this for each $m \in M$.

The two ways coincide.

I claim that these two ways of getting vector fields coincide; that is, I claim that

$$X_{\mathcal{H}}(m) = \left[\mathcal{L}_F\left(\Phi_M(m)\right)\right]_M(m) \quad \forall\, m \in M. \tag{2}$$

To prove (2) it suffices to show that both sides of (2) give the same value when we take their symplectic scalar product (using $(\omega_M)_m$) with any $v \in T_m M$.

Since $i(X_{\mathcal{H}})\omega_M = d\mathcal{H} = d(F \circ \Phi_M)$, the symplectic scalar product $\omega_M(X_{\mathcal{H}}(m), v)_m$ is just the directional derivative

$$v(\mathcal{H}) = v(F \circ \Phi_M) = (d(\Phi_M)_m(v))F$$

by the chain rule.

Proof.

To prove: $$X_{\mathcal{H}}(m) = \left[\mathcal{L}_F\left(\Phi_M(m)\right)\right]_M(m) \quad \forall\ m \in M. \tag{2}$$

Since $i(X_{\mathcal{H}})\omega_M = d\mathcal{H} = d(F \circ \Phi_M)$, $\omega_M(X_{\mathcal{H}}(m), v)_m$ is

$$v(\mathcal{H}) = v(F \circ \Phi_M) = (d(\Phi_M)_m(v))F$$

By the definition of $\mathcal{L}_F$ this is just

$$\langle d(\Phi_M)_m(v), \mathcal{L}_F(\Phi_M(m))\rangle.$$

We now recall the formula for the derivative of the moment map. Writing Φ for Φ_M and ω for ω_M this says that

$$\langle d\Phi_m(v), A\rangle = \omega_m(A_M(m), v).$$

Applied to $A = \mathcal{L}_F\left(\Phi_M(m)\right)$ this shows that

$$\langle d\Phi_m(v), \mathcal{L}_F(\Phi(m))\rangle = \omega_m\left(\left[\mathcal{L}_F\left(\Phi(m)\right)\right]_M(m), v\right). \quad \square$$

Consequences.

$$X_{\mathcal{H}}(m) = \left[\mathcal{L}_F\left(\Phi_M(m)\right)\right]_M(m) \quad \forall\, m \in M. \tag{2}$$

It follows from (2) and the equivariance of Φ_M that

$$d(\Phi_M)_m(X_{\mathcal{H}}(m)) = [\mathcal{L}_F(\Phi_M(m))]_{\mathfrak{g}^*}(\Phi_M(m)) \tag{3}$$

where the right hand side is the value at $\Phi_M(m)$ of the generating vector field given by the coadjoint action associated to element $\mathcal{L}_F(\Phi_M(m)) \in \mathfrak{g}$. But this is just the value at $\Phi_M(m)$ of the vector field associated to the function F when restricted to the orbit $\mathcal{O}$ through $\Phi_M(m)$. (This can either be seen directly or by applying (2) to the orbit $\mathcal{O}$ thought of as a symplectic G-space whose moment map is the injection of $\mathcal{O}$ into $\mathfrak{g}^*$.)

Thus, if $m(t)$ denotes the trajectory of the Hamiltonian system $X_{\mathcal{H}}$ with $m(0) = m$, we see that $\Phi_M(m(t))$ lies entirely on the orbit $\mathcal{O}$ through $\Phi_M(m)$ and is a solution curve of the Hamiltonian system corresponding to F.

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Let $\gamma(t)$ denotes this curve in $\mathcal{O}$. We can form the curve

$$A(t) = \mathcal{L}_F(\gamma(t))$$

in $\mathfrak{g}$ and (2) says that

$$m'(t) = [A(t)]_M(m(t)).$$

So we can find the solution curve $m(t)$ to $X_{\mathcal{H}}$ by applying the following four steps:

Solving the equations in four easy steps.

So we can find the solution curve $m(t)$ to $X_{\mathcal{H}}$ by applying the following four steps:

1. Find the orbit $\mathcal{O}$ through $\Phi_M(m)$.

2. Find the solution curve to the Hamiltonian system on $\mathcal{O}$ corresponding to F, passing through $\Phi_M(m)$ at $t = 0$. Call this curve $\gamma(t)$.

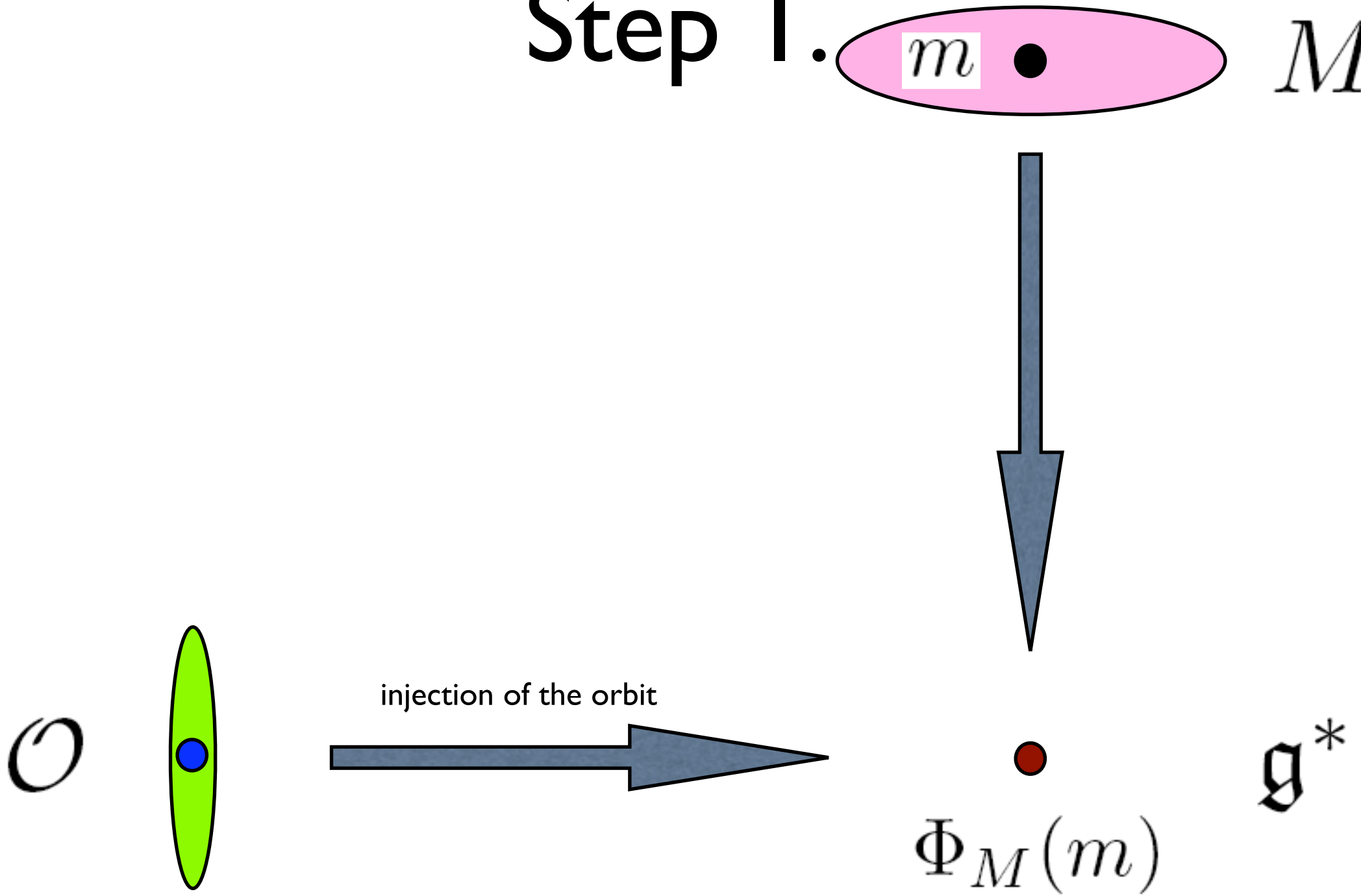
3. Compute the curve $A(t) = \mathcal{L}_F(\gamma(t))$ This is a curve in $\mathfrak{g}$.

4. Solve the differential equations (i.e., find the curve in G satisfying),

$$a'(t) = -A(t)a(t), \quad a(0) = e.$$

Then $a(t)m$ is the desired solution curve.

Step 1.



Step 1 is purely kinematic; it depends solely on the Hamiltonian group action and has nothing to do with F.

Step 2.

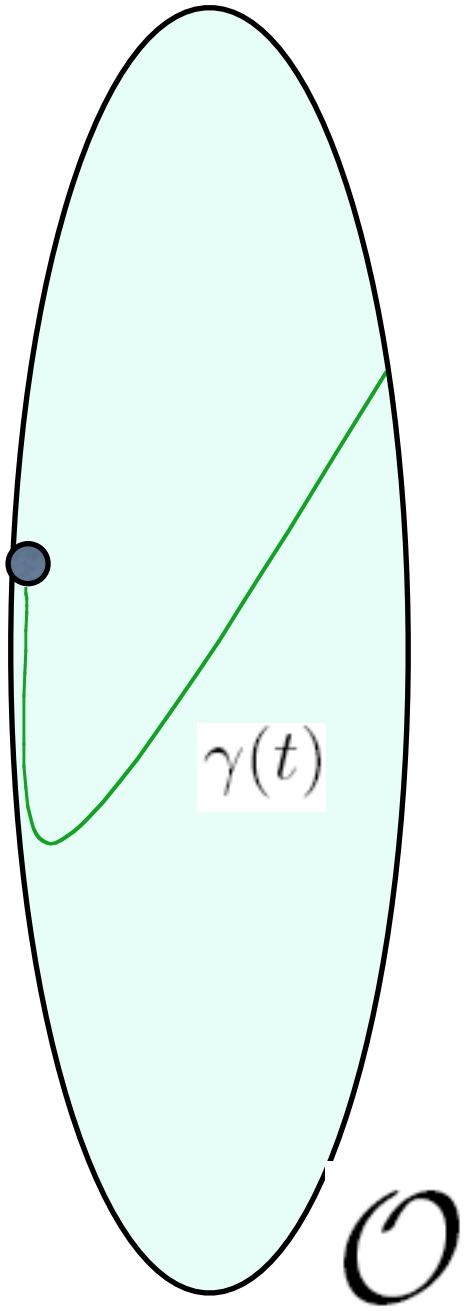
Step 2 involves solving a Hamiltonian system with (usually) many fewer degrees of freedom than M. Thus, for example, in the liquid drop model, $\mathcal{O}$ is at most 12-dimensional, while $M = \mathbb{R}^{6N}$ has dimension $6N$. This type of Hamiltonian equation has become popular in recent years in the study of mechanical systems associated with nonlinear partial differential equations. In case $\mathfrak{g}$ has a nondegenerate invariant bilinear form (which is definitely not the case for the semidirect product groups above), one can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ so that $\mathcal{O}$ becomes an orbit in $\mathfrak{g}$ and the differential equations become

$$\frac{d\gamma}{dt} = [\gamma(t), \mathcal{L}_F(\gamma(t))]$$

which are known as Lax equations

If $G = Sl(n)$ of $U(n)$ the fact that $\gamma(t)$ lies on a fixed orbit means that the eigenvalues of the $\gamma(t)$ remain constant, so one speaks of an "isospectral deformation." However, the natural setting is on an orbit in $\mathfrak{g}^*$ not $\mathfrak{g}$.

Step 2.



2. Find the solution curve to the Hamiltonian system on $\mathcal{O}$ corresponding to F, passing through $\Phi_M(m)$ at $t = 0$. Call this curve $\gamma(t)$.

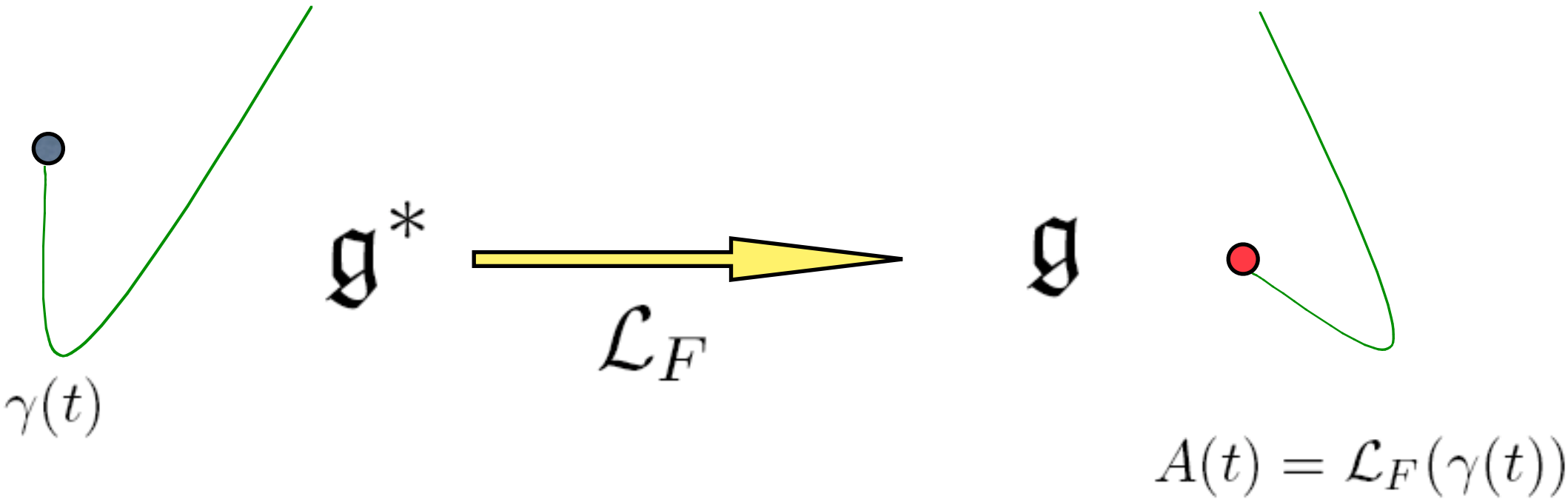
Steps 3 and 4.

Step 3 is an application of the Legendre transformation.

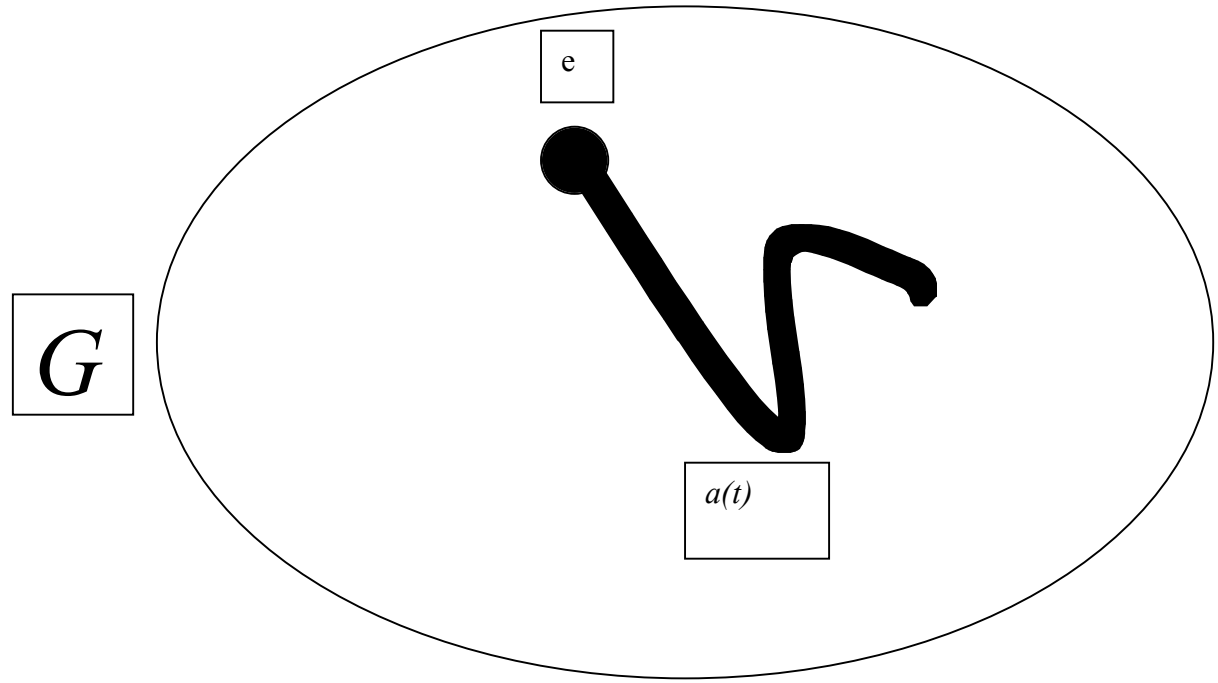
Step 4 can pose some interesting problems even if the solution of step 2 is trivial. For instance, suppose that F is a G-invariant. Then on each $\mathcal{O}$, the curve $\gamma(t)$ is a constant, but the map $\mathcal{L}_F$ need not be trivial. Thus, $A(t)$ will be a constant element of $\mathfrak{g}$ and so $a(t)$ will be a one-parameter group. Thus the motion corresponding to $F \circ \Phi_M$ when F is an invariant is given by the action of a one-parameter group, the one-parameter group depending on m. (For the case of a spherical top, this is the spinning motion.) We might think of the solutions for noninvariant F as "generalized precessions or nutations."

Step 3.

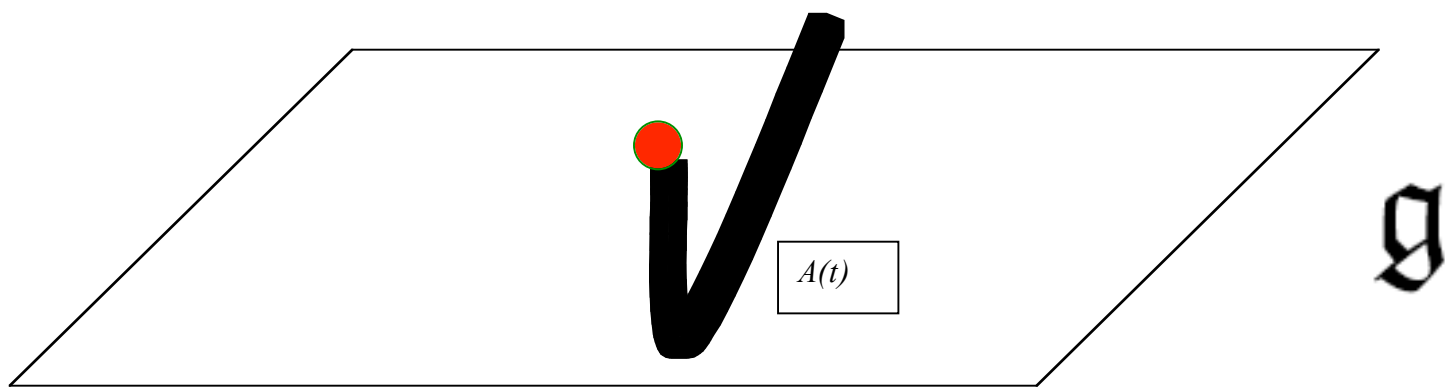
Step 3 is an application of the Legendre transformation.



Step 4.



$$a'(t) = -A(t)a(t), \quad a(0) = e.$$



Partial information.

Notice that step 4 simplifies if we are only interested in partial information about the trajectory $m(t)$. For example, suppose that $M = T^*Q$ and $m = (q, p)$, where $q \in Q$. That is, m is a point in phase space whose corresponding point in configuration space is q. We might only be interested in the time evolution of q, and this may involve less than the full curve $a(t)$. In the case of the groups for the rigid body or liquid drop models it is only the $SO(3)$ or $Sl(3)$ component that acts on configuration space. The curve$A(t) \in \mathfrak{g}$ can be written as $A(t) = (B(t), C(t))$, where $B(t) \in o(3)$ or $sl(3)$. Then $b(t)$, the $H = SO(3)$ or $Sl(3)$ component of $a(t)$ can be found by solving the equation $b'(t) = -B(t)b(t)$ in H.

The Poisson structure on the dual of a Lie algebra.

We can use the Legendre transformation to make the space of smooth functions (or polynomials) defined on (an open subset of) $\mathfrak{g}*$ into a Poisson algebra. We simply define

$$\{f_1, f_2\}(\mu) := \langle \mu, [\mathcal{L}_{f_1}(\mu), \mathcal{L}_{f_2}(\mu)] \rangle \tag{3}$$

for any μ in the common domain of definition of f_1 and f_2. The right hand side of (3) makes sense since $\mathcal{L}_{f_1}(\mu)$ and $\mathcal{L}_{f_2}(\mu)$ are elements of $\mathfrak{g}$ and so we can compute their Lie bracket. We can then evaluate μ on this bracket.

Relation to the symplectic structure on orbits

$$\{f_1, f_2\}(\mu) := \langle \mu, [\mathcal{L}_{f_1}(\mu), \mathcal{L}_{f_2}(\mu)] \rangle \tag{3}$$

In view of the discussion leading to the proof of (2) we know that

$$(\{f_1, f_2\})_{\mathcal{O}} = \left\{ f_{1_{|\mathcal{O}}}, f_{2_{|\mathcal{O}}} \right\}_{|\mathcal{O}} \tag{4}$$

where the bracket on the right is the Poisson bracket coming from the symplectic structure of $\mathcal{O}$. This proves that (3) does indeed define a Poisson bracket, that is, satisfies the axioms for a Poisson bracket.

It also shows that $\mathfrak{g}^*$ as a Poisson manifold is "stratified" into a union of the symplectic manifolds consisting of the co-adjoint orbits.

The advantage to the definition (3) is that it is purely Lie algebra based, and does not involve the group-theoretical notion of orbit. Hence, it is of use in certain infinite-dimensional situations where the group-theoretical constructions are not available.

Pull back by the moment map is a homomorphism of Poisson algebras.

Let (M, ω_M, Φ_M) be a Hamiltonian G-space. We can consider the corresponding map Φ_M^* from functions on $\mathfrak{g}^*$ to functions on M,

$$\Phi_M^* : C^\infty(\mathfrak{g}^*) \to C^\infty(M) : \quad f \mapsto \Phi_M^* f = f \circ \Phi_M$$

which assigns to each function f on $\mathfrak{g}^*$ the corresponding "collective Hamiltonian" on M. From the above discussion we know that is a homomorphism of Poisson structures:

$$\phi_M^*(\{f_1, f_2\}) = \{\Phi_M^*(f_1), \Phi_M^*(f_2)\}_M. \tag{6}$$

Collective and invariant Hamiltonians commute.

Suppose that $f \in C^\infty(M)$ is a G-invariant function. We write this as $f \in C^\infty(M)^G$. If $T_m M \ni v \in \mathfrak{g}_M(m)$ then $(vf)(m) = 0$.

So if $\mathcal{H}$ is a collective Hamiltonian, $X_{\mathcal{H}} f = 0$, i.e.

$$\{H, f\} = 0.$$

The collective Hamiltonians and the invariant Hamiltonians Poisson commute with one another.

Do they mutually centralize one another?

At any $m \in M$, the subspace $\mathfrak{g}_M(m)^\perp \subset T_m M$ is spanned by those covectors which are of the form $df(m)$ where $A_M f(m) = 0$ for all $A \in \mathfrak{g}$. Since $(\mathfrak{g}_M(m)^\perp)^\perp = \mathfrak{g}$, this suggests that the collective Hamiltonians and the invariant Hamiltonins mutually centralize one another, i.e that the collective Hamiltonians are all the Hamiltonians which Poisson commute with all the invariant Hamiltonians and vice versa. This is not quite true without some additional assumptions which we will discuss.

But first an interesting example.

Left and right actions of G on its cotangent bundle.

Recall that the moment map for the right action of G on T^*G is

$$\Phi^R(g, \mu) = \mu$$

under the left identification of T^*G with $G \times \mathfrak{g}^*$. Under this identification, the left action of $a \in G$ is given by $a(g\mu) = (ag, \mu)$. So we see that every collective Hamiltonian for the right action is invariant for the left action. As to the converse, any invariant function for the left action is the pull-back of a function on $\mathfrak{g}^*$. What has to be checked is whether or not this function is a smooth function on $\mathfrak{g}^*$. We will do this in more generality later on. Of course we may interchange right and left so

Proposition 1 *Every right collective Hamiltonian is left invariant and every left collective Hamiltonian is right invariant.*

Example: the free rigid body.

Let us show how Proposition 1, together with the integration procedure described earlier, gives a prescription for solving the equations of motion of a free rigid body. For the rigid body, the configuration space is taken to be $SO(3)$, the group of rotations of the body about its center of mass.

The left-invariant Hamiltonian in this case is given by

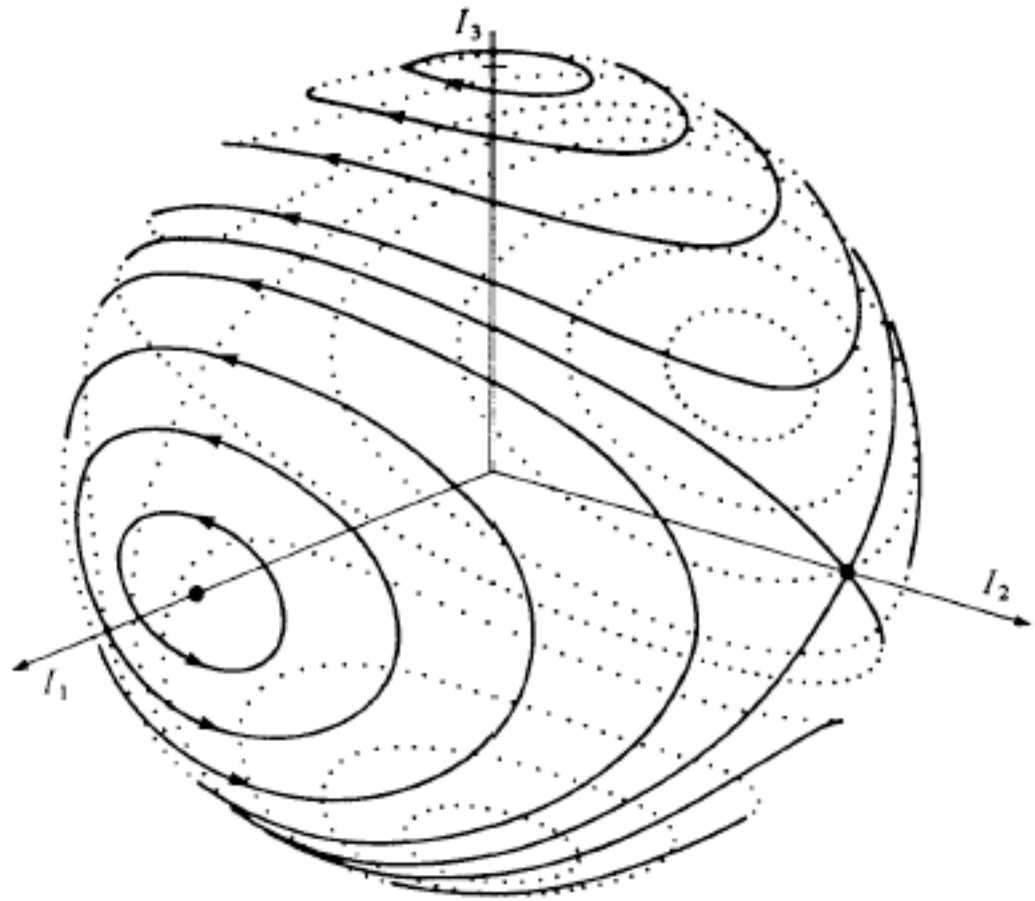
$$\mathcal{H}(g,\mu) = \mathcal{H}(\mu) = \frac{1}{2}\sum_{i=1}^{3}\frac{\mu_i^2}{I_i}$$

where the I_i are the "moments of inertia," and we have chosen a basis in which the inertia tensor is diagonal.

Example: the free rigid body, 2.

$$\mathcal{H}(g,\mu) = \mathcal{H}(\mu) = \frac{1}{2}\sum_{i=1}^{3}\frac{\mu_i^2}{I_i}$$

This is collective for the right action. The $SO(3)$ co-adjoint orbits are spheres, and the problem of integrating the Hamiltonian system on each orbit (step 2 above) becomes easy since the flow lines must be level curves of $\mathcal{H}$, so that the flow lines are obtained by intersecting the ellipsoids $\mathcal{H}$ = constant with the spheres. The radius of the sphere is called the total angular momentum. For distinct moments of inertia $I_1 > I_2 > I_3$ the flow on the sphere of radius m has saddle points at $(0, \pm m, 0)$ and centers at $(\pm m, 0, 0)$ and $(0, 0, *m)$, corresponding to the critical points of $\mathcal{H}$, restricted to the sphere. The saddles are connected by four "heteroclinic" orbits as indicated in the figure.



These curves tell us how the instantaneous "axis of rotation" is changing: We must apply step 3, which gives us (in this case) a linear map from g^* to g (sending $(\alpha_1, \alpha_2, \alpha_3)$ into $(\alpha_1/I_1, \alpha_2/I_2, \alpha_3/I_3)$), which, when applied to any orbit on the sphere, gives the instantaneous rotation. The actual motion of the rigid body is then obtained by applying step 3 above.

The rest of this lecture is taken from the paper "The centralizer of invariant functions and the division properties of moment maps" by Karshon and Lerman *Ill. J. Math.* **41**, no. 3 (1997), 462-487.

The centralizer of the invariant functions.

Let (M, ω, Φ) be a Hamiltonian G-space where G is a *compact* Lie group.

Theorem 1 *The centralizer of the G-invariant functions functions is the set of smooth functions which are locally constant on every level set of Φ.*

Proof. We have already remarked that the Hamiltonian flow of an invariant function preserves the level sets of Φ. So the Poisson bracket of an invariant function and a function which is locally constant on the level sets of Φ is zero. So the centralizer of the invariant functions contains the functions which are locally constant on the level sets of Φ. We need to show that there is nothing else in the centralizer.

So let h be a smooth function in this centralizer. Let $\gamma(t)$ be a smooth curve contained in a level set of Φ. Since any two points of every connected component can be joined by a piecewise smooth curve, it is enough to show that the derivative of $h(\gamma(t))$ is zero for all t. This derivative is equal to $\omega(X_h, \dot{\gamma})$. For any $A \in \mathfrak{g}$ we have

$$0 = d\Phi^A(\dot{\gamma}) = \omega(A_M, \dot{\gamma}).$$

So the $\dot{\gamma}(t)$ lie in the symplectic perpendiculars to the G-orbits. So it is enough to show that X_h is tangent to the G-orbits.

Let $t \mapsto \sigma(t)$ be an integral curve of X_h. Let f be a G-invariant function. By assumption $X_h f = 0$. But

$$\frac{d}{dt}(f(\sigma(t)) = (X_h f)(\sigma(t))$$

and so equals 0. In other words, f is constant along $\sigma(t)$.

Since G is compact, the G-orbits are compact, and hence two distinct orbits are contained in two disjoint open sets. Hence the G-invariant functions separate orbits. Hence σ lies in a fixed orbit, and so X_h is tangent to the G-orbits. $\square$

By the Jacobi identity, the centralizer of any subset of $C^\infty(M)$ is a Poisson subalgebra. So a corollary of our theorem is

Corollary 1 *The set of smooth functions which are locally constant on G-orbits is a Poisson subalgebra of $C^\infty(M)$.*

If a function f Poisson commutes with all the Φ^A, $A \in \mathfrak{g}$ then $X_A f = 0$ for all $A \in \mathfrak{g}$ which implies that f is constant under the action of G^0, the connected component of the identity in G. Of course, it is enough to check this for A ranging over a basis of $\mathfrak{g}$. So we have

Proposition 2 *If $f \in C^\infty(M)$ Poisson commutes with all the Φ^A as A ranges over a basis of $\mathfrak{g}$ then $f \in C^\infty(M)^{G^0}$.*

Three types of "collectives".

Let $C^\infty(M)^\Phi$ denote the smooth functions which are constant on the level sets of the moment map. Let $C^\infty(M)^\Phi_{loc}$ denote the set of functions which are locally constant on the level sets of the moment map and let $\Phi^*(C^\infty(\mathfrak{g}^*))$ denote the collective functions, that is the pull-back of smooth functions on $\mathfrak{g}^*$ via the moment map. So

$$\Phi^*(C^\infty(\mathfrak{g}^*)) \subset C^\infty(M)^\Phi \subset C^\infty(M)^\Phi_{loc}. \tag{7}$$

In this notation, Theorem 1 says that the centralizer of $C^\infty(M)^G$ is $C^\infty(M)^\Phi_{loc}$.

On the other hand, Prop.2 implies that the centralizer of the collective functions is $C^\infty(M)^{G^0}$. So we know that

Corollary 2 *The subalgebras $C^\infty(M)^\Phi_{loc}$ and $C^\infty(M)^{G^0}$ are mutual centralizers.*

Reformulation of the problem.

$$\Phi^*(C^\infty(\mathfrak{g}^*)) \subset C^\infty(M)^\Phi \subset C^\infty(M)^\Phi_{loc}. \qquad (7)$$

So if G is connected, the issue to be studied is when are the three spaces in (7) equal. For this I refer you to the paper by Karshon and Lerman.