Symplectic geometry
Lecture 15

Semi-direct products,
their co-adjoint orbits and
their moment maps.

The semi-direct product of a group with a vector space.

Let H be a Lie group and suppose we are given a representation of H on a real vector space V. We will denote the image of the action of an element $a \in H$ on a $v \in V$ by av. We construct the **semi-direct** product G of H with V as follows: As a set, $G = H \times V$ endowed with the multiplication:

$$(a, v) \cdot (b, w) := (ab, aw + v).$$

The “matrix” mnemonic.

An easy way to remember this multiplication is to write the element (a, v) as a “matrix”

$$\begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix}.$$

Then the multiplication in G is just the “matrix multiplication”

$$\begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & w \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ab & aw + v \\ 0 & 1 \end{pmatrix}.$$

Similarly, we will write an element of the Lie algebra $\mathfrak{g}$ of G as a “matrix”

$$\begin{pmatrix} A & x \\ 0 & 0 \end{pmatrix}, \quad A \in \mathfrak{h}, \quad x \in V$$

where $\mathfrak{h}$ is the Lie algebra of H.

The co-adjoint representation.

Let us write the most general element of $\mathfrak{g}^*$ as (α, p) where $\alpha \in \mathfrak{h}^*$ and $p \in V^*$ so that

$$\left\langle (\alpha, p), \begin{pmatrix} A & x \\ 0 & 0 \end{pmatrix} \right\rangle = \langle \alpha, A \rangle + \langle p, x \rangle.$$

Let us denote the co-adjoint representation by $\mathrm{Ad}^\sharp$. So

$$\mathrm{Ad}^\sharp_g := \mathrm{Ad}^*_{g^{-1}} .$$

Then

$$\begin{aligned}\left\langle [\mathrm{Ad}^{\#}_{\begin{pmatrix} a & v \\ 0 & 1\end{pmatrix}}(\alpha, p)], \begin{pmatrix} A & x \\ 0 & 0 \end{pmatrix}\right\rangle &= \left\langle (\alpha, p), \mathrm{Ad}_{\begin{pmatrix} a^{-1} & -a^{-1}v \\ 0 & 1\end{pmatrix}}\begin{pmatrix} A & x \\ 0 & 0 \end{pmatrix}\right\rangle \\ &= \left\langle (\alpha, p), \begin{pmatrix} a^{-1}Aa & a^{-1}x + a^{-1}Av \\ 0 & 0 \end{pmatrix}\right\rangle \\ &= \langle \alpha, a^{-1}Aa\rangle + \langle p, a^{-1}Av\rangle + \langle p, a^{-1}x\rangle.\end{aligned}$$

Define a map

$$V^* \times V \to \mathfrak{h}^*, \quad (p, v) \mapsto p \odot v$$

by

$$\langle p \odot v, A \rangle := \langle p, Av \rangle.$$

Let $\mathrm{Ad}^{\sharp\mathfrak{h}}$ denote the coadjoint action of H on $\mathfrak{h}^*$. Then we have proved that

$$\mathrm{Ad}^{\sharp}_{\begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix}}(\alpha, p) = \left(\mathrm{Ad}^{\sharp\mathfrak{h}}_{a}\, \alpha + a^{*-1} p \odot v, a^{*-1} p \right). \tag{4}$$

The coadjoint G orbits in $\mathfrak{g}^*$ are thus fibered over the H orbits in V^*.

Describing the fibration.

Suppose we pick an H orbit N in V^* and want to describe the various G orbits in $\mathfrak{g}^*$ "sitting over N": Pick some point $p \in N$ and let $H_p \subset H$ be the isotropy group of p. Let $\mathfrak{h}_p$ be the Lie algebra of H_p. Since $\mathfrak{h}_p$ is a sub-algebra of $\mathfrak{h}$ we get a projection

$$\pi_p : \mathfrak{h}^* \to \mathfrak{h}_p^*$$

where $\pi_p \alpha$ is simply the restriction of α which is a linear function on all of $\mathfrak{h}$ to the subspace $\mathfrak{h}_p$. We claim that

$$\alpha - \alpha^0 = p \odot v \quad \text{for some} \quad v \in V$$

if and only if

$$\pi_p \alpha = \pi_p \alpha^0.$$

Describing the fibration, 2.

Indeed, if $A \in \mathfrak{h}$ then

$$\langle p \odot v, A \rangle = \langle p, Av \rangle = \langle A^* p, v \rangle.$$

If $A \in \mathfrak{h}_p$ so that $A^* p = 0$, this implies that $\langle p \odot v, A \rangle = 0$. Since this is true for all $A \in \mathfrak{h}_p$ we see that $\pi_p \alpha = \pi_p \alpha^0$.

Describing the fibration, 3.

Now for the converse: We may assume that $p \neq 0$. Then the set of all elements of $gl(V) = V \otimes V^*$ that are orthogonal to all $p \otimes v,\ \ v \in V$ is $(n^2 - n)$-dimensional. It contains the $(n^2 - n)$-dimensional space consisting of all $B \in gl(V)$ satisfying $B^* p = 0$ by the preceding computation applied to $gl(V)$. Thus these two spaces coincide. The image of $p \otimes V$ in $\mathfrak{h}^*$ under the map $V^* \otimes V \to \mathfrak{h}^*$ dual to the map $\mathfrak{h} \to gl(V) = V \otimes V^*$ is $p \odot V$. From this it follows that the annihilator space of $p \odot V$ in $\mathfrak{h}$ is just the inverse image in $\mathfrak{h}$ of $gl(V)_p$, that is $\mathfrak{h}_p$. $\square$

Thus, once we have picked an orbit N of H acting on V^* and a point $p \in N$, the classification of the orbits above N reduce to the classification of the H_p orbits in $\mathfrak{h}_p^*$.

“Euclidean” groups.

Thus, once we have picked an orbit N of H acting on V^* and a point $p \in N$, the classification of the orbits above N reduce to the classification of the H_p orbits in $\mathfrak{h}_p^*$.

Let us see what these results say when V is a vector space equipped with a nondegenerate scalar product and $H = SO(V)$ is the connected component of the orthogonal group for this scalar product.

As sub-examples we could take $V = \mathbb{R}^3$ with its standard (positive definite) scalar product so $H = SO(3)$ is the rotation group in three dimensions and G is then the group of Euclidean motions in three dimensions. Or we could take V to be Minkowski space $\mathbb{R}^{1,3}$ so that $H = L$ is the connected component of the Lorentz group and G is the Poincaré group.

Some identifications.

Back to the general study of the case where V is a vector space equipped with a nondegenerate scalar product and $H = SO(V)$ is the connected component of the orthogonal group for this scalar product: We may identify V with V^* using the scalar product, and also identify $\mathfrak{h}$ with $\mathfrak{h}^*$. We may also identify $\mathfrak{h}$ with $\bigwedge^2(V)$ where $u \wedge v$ is identified with the operator $A_{u\wedge v}$ where

$$A_{u\wedge v} w := (w, v)u - (v, u)w$$

and where $(\ , \ )$ is the scalar product on V.

Let us examine the meaning the operator

$$V^* \times V \to \mathfrak{h}^*, \quad (p, v) \mapsto p \odot v$$

defined by

$$\langle p \odot v, A\rangle := \langle p, Av\rangle.$$

under all these identifications. We have

$$\begin{aligned}\langle p, A_{u\wedge w}v\rangle &= \big(p, (w,v)u - (v,u)w\big)\\ &= (p,u)(w,v) - (p,w)(v,u)\\ &= \det\begin{pmatrix}(p,u) & (p,w)\\ (v,u) & (v,w)\end{pmatrix}.\end{aligned}$$

This determinant is precisely the scalar product of the elements $p\wedge v$ and $u\wedge w$ in $\bigwedge^2(V)$ with respect to the scalar product induced from the scalar product on V. If we denote this by $(\ ,\ )$ as well, then our definition of $p\odot v$ becomes

$$\langle p\odot v, A_{v\wedge w}\rangle = (p\wedge v, u\wedge w)$$

or, simply

$$p\odot v = p\wedge v.$$

The co-adjoint action.

There is also some slight simplification in the description of the co-adjoint representation (4):

$$\mathrm{Ad}^\sharp_{\begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix}}(\alpha, p) = \left(\mathrm{Ad}^{\sharp\flat}_a \alpha + a^{*-1} p \odot v, a^{*-1} p\right).$$

Under the identification of $\bigwedge^2(V)$ with $o(V)^*$, the coadjoint action of $SO(V)$ becomes identified with the usual action of $SO(V)$ on $\wedge^2(V)$. Also, under the identification of V with V^*, a^{*-1} becomes simply a. So

$$\mathrm{Ad}^\sharp_{\begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix}}(\alpha, p) = (a\alpha + ap \wedge v, ap).$$

$$\mathrm{Ad}^\sharp_{\begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix}}(\alpha, p) = (a\alpha + ap \wedge v, ap).$$

In other words, the rotation a acts on (α, p) by simply rotating α and p. The translation through v does not change p but does replace α by $\alpha + p \wedge v$.

The orbits of SO(V) acting on V are “spheres”.

The orbits of $SO(V)$ acting on V are the connected components of the (dim V - 1) dimensional “spheres” $\|p\|^2 = \text{const.}, p \neq 0$ and the zero-dimensional orbit, consisting of the origin $\{0\}$.

The case of E(3).

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For the case of $E(3)$ these orbits of $SO(3)$ acting on $\mathbb{R}^3$ are just two dimensional spheres (or the origin). So, for example, we may apply a suitable rotation to the element (β, p) to bring it to the form (γ, ke_3), where e_3 denotes the unit vector in the z direction and $k = \|p\|$. There are now two alternatives: $k = 0$ or $k \neq 0$.

The zero and two dimensional orbits of E(3).

If $k = 0$ so that $p = 0$, then $p \wedge v = 0$ so that the action of the translations on $(\gamma, 0)$ is trivial, and the action of a rotation a is to send $(\gamma, 0)$ to $(a\gamma, 0)$. Again this subdivides into two cases: $\gamma = 0$ and $\gamma \neq 0$.

If $\gamma = 0$ then we get the zero-dimensional coadjoint orbit $(0, 0)$. If $\gamma \neq 0$ then the coadjoint orbit through γ is the two-dimensional sphere. These are the zero and two dimensional co-adjoint orbits of $E(3)$.

The four dimensional co-adjoint orbits of E(3).

Now let us turn to the case where $p \neq 0$. If $v = ae_1 + be_2 + ce_3$ then

$$p \wedge v = kae_3 \wedge e_1 + kbe_3 \wedge e_2.$$

Writing

$$\gamma = xe_1 \wedge e_3 + ye_2 \wedge e_3 + se_1 \wedge e_2$$

we can uniquely choose a and b so that

$$\gamma + p \wedge v = se_1 \wedge e_2.$$

In other words, on every orbit with $p \neq 0$, there is a unique point of the form

$$\mu = (se_1 \wedge e_2, ke_3).$$

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It is clear that the subgroup of $E(3)$ that leaves μ fixed is precisely the group of those Eucridean motions that fix the z-axis. Thus, as homogeneous $E(3)$ spaces, all these orbits look like the space of straight lines in $\mathbb{R}^3$. But the different symplectic structures are parameterized by the real number s and the positive number k. For μ as above, we have

$$(se_1 \wedge e_2) \wedge ke_3 = (sk)e_1 \wedge e_2 \wedge e_3.$$

For general (β, p) we have

$$\beta \wedge p = (a\beta + ap \wedge v) \wedge ap \quad \forall\ v \in V \quad a \in SO(3)$$

since $ap \wedge v \wedge ap = 0$ and $a\beta \wedge ap = (\det a)\beta \wedge p$ and $\det a = 1$.

The invariants *k* and *s* .

So the functions k defined by

$$k(\beta, p) = \|p\|$$

and s defined for those (β, p) with $p \neq 0$ by

$$\beta \wedge p = ske_1 \wedge e_2 \wedge 3_3$$

are are invariant under the action of $E(3)$ and the four-dimensional orbits are described by the different possible values of the functions k and s.

The moment map for E(3) acting on six-dimensional phase space.

So we are examining the phase space of a single particle. The action of

$$\begin{pmatrix} a & w \\ 0 & 1 \end{pmatrix}$$

is realized as a Euclidean motion by matrix multiplication:

$$\begin{pmatrix} a & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} Aq + w \\ 1 \end{pmatrix}.$$

So

$$\exp -t \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} q - t(Ax + v) + \cdots \\ 1 \end{pmatrix}$$

and so the fundamental vector field corresponding to $\begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix}$ is

$$q \mapsto -Aq - v.$$

Explicit form of the moment map.

Thus

$$\left\langle \Phi(q,p), \begin{pmatrix} A & v \\ 0 & 0 \end{pmatrix} \right\rangle = -p \cdot Aq - p \cdot v$$

or

$$\Phi(q,p) = -(p \wedge q, p). \tag{5}$$

So if $p \neq 0$ the image of (q,p) is the four dimensional co-adjoint orbit with $k = \|p\|$ and $s = 0$. (The image of a point of the form $(0,q)$ is the zero dimensional co-adjoint orbit consisting of the origin.)

The inverse image of the moment map.

$$\Phi(q,p) = -(p \wedge q, p). \tag{5}$$

If $p \neq 0$, two points (q,p) and (q',p') have the same image under Φ if and only if $p' = p$ and $p \wedge q' := p \wedge q$. This last condition can hold if and only if $q - q'$ is some multiple of p. Thus

$$\Phi^{-1}\left(\Phi(q,p)\right) = \{q + tp, p)\}$$

is a straight line, which we may think of as the trajectory of the particle with momentum p. In this sense we see that the four-dimensional symplectic orbits of $E(3)$ with $s = 0$ correspond to the space of trajectories of classical particles; the straight lines in our homogeneous space description of the coadjoint orbit are interpreted as the trajectories of the particle.

Intrinsic angular momentum.

We know that $s = 0$ because a standard classical particle has no "intrinsic angular momentum." The existence of other coadjoint orbits with $s \neq 0$ suggests that there should exist particles with an "intrinsic" angular momentum or "spin." In fact, in an experiment done in the 1930s by Beck at the suggestion of Einstein, the photon was shown to have just such an intrinsic angular momentum measured mechanically.

Units.

We now come to another important point. If we think of q as having units of length, then k is an object having units of inverse length. Indeed, this is the natural choice of units from the group-theoretical point of view, since in $e(3)^*$, p is something dual to translations. But in classical mechanics, p of k is described in terms of "momemtum". What is the relation between the two? Let me try to give an explanation in the form of historical science fiction:

Historical science fiction.

Suppose that mechanics had developed before the invention of clocks. So we could observe the trajectories of particles, their collisions and deflections, but not their velocities. For instance we might be able to observe tracks in a bubble chamber or on a photographic plate. (In the case of light, all of the work described above could have done before there was any accurate measurement of the velocity of light.) The configuration space of a single particle is just the three-dimensional Euclidean space $\mathbb{R}^3$, the corresponding phase space is six-dimensional with coordinates (q, p), where q and p are 3-vectors. As we have seen, each free classical particle; that is, each of the five-dimensional orbits of $E(3)$ acting on the six-dimensional phase space is parametrized by its value of $k = \|p\|$. In the absence of clocks we cannot measure velocity so we cannot distinguish between a "light particle moving fast" or a "heavy particle moving slowly." (Of course, from the scattering experiments themselves, we would be led to discover new conserved quantities such as energy, and thus be led to enlarge the group. But we do not want to go into this point.)

The classical meaning of Planck's constant.

Without some way of relating momentum to length, we would introduce "independent units" of momentum, perhaps by combining particles in various ways and performing collision experiments. But we know that the "natural units" should be inverse length. A single experiment. the photoelectric effect, involving an interaction between light and one of our "particles" would then give us the conversion factor, and allow us to write $\|p\| = h/\lambda$. Thus, from this group-theoretical point of view, Planck's constant h is a conversion factor from the "independent" units of momentum to the "natural" units of inverse length. Of course, the story did not develop that way. The "conversion factor" was first found between "energy" and "inverse time"

Co-adjoint orbits of the Poincare group.

By our general theory we know that we can describe the dual of the Lie algebra as

$$\wedge^2 V \oplus V$$

where $V = \mathbb{R}^{1,3}$. We also know that if we write an element of this ten dimensional space as (Γ, P) then

$$\|P\|^2 \quad \text{and} \quad \|\Gamma \wedge P\|^2$$

are invariants. The first step is to describe the orbits of $SO(1,3)$ acting on V^*. The function $\|P\|^2$ is invariant, and so each orbit must lie on a level set of this function. If we choose a space-time splitting in terms of which $P = (E, p)$ we can write

$$\|P\|^2 = E^2 - c^2 p^2$$

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where c is the speed of light and p^2 denotes the square length of the three vector p. So there are six types of $SO(1,3)$ acting on V^*:

The six types of orbits of the Lorentz group:

1. Orbits with $\|P\|^2 > 0$ and $E > 0$.
2. Orbits with $\|P\|^2 > 0$ and $E < 0$.
3. The orbit with $\|P\|^2 = 0$ and $E > 0$.
4. The orbit with $\|P\|^2 = 0$ and $E > 0$
5. Orbits with $\|P\|^2 < 0$.
6. The single point orbit $\{0\}$.

The rest mass.

For orbits of type 1) and 2) we may write

$$\|P\|^2 = m^2c^4$$

where m is called the "rest mass". Then orbits of type 3) and 4) correspond to particles of mass zero, while of type 5) correspond to "tachyons " which are not believed to exist.

The positive mass co-adjoint orbits.

Let us pass to units where $c = 1$ and examine the co-adjoint orbits corresponding to the V orbits of type 1). On such a V orbit there is a unique point of the form $(E, 0)$. The subgroup of $SO(1,3)$ fixing this point is $SO(3)$. The co-adjoint orbits for $SO(3)$ are just spheres of various radius and the origin. By our general method, we can subtract off $P \wedge Q$ form Γ so as to arrange that $\Gamma \in \bigwedge^2(\mathbb{R}^3)$ and so we see that

$$\|\Gamma \wedge P\|^2 = m^2 s^2$$

where s is the radius of the sphere describing the co-adjoint orbits of $SO(3)$. the parameter s is called the "‘spin". So corresponding to orbits of type 1) of $SO(1,3)$ acting on V^* there are two types of co-adjoint orbits:

Mass and spin.

- Eight dimensional orbits with $m > 0$ and $s > 0$. These are positive mass particles with intrinsic spin s.
- Six dimensional orbits with $m > 0$ and $s = 0$. these are massive particles with no intrinsic spin.

Notice, in contrast to what is told in physics courses - that spin is a purely quantum effect - we see that spin arises naturally in the classification of the co-adjoint orbits of the Poincare group.

Quadrupole moments.

I want to consider the case of a semi-direct product G where H is a subgroup of $Gl(n)$ and $V = S^2(\mathbb{R}^{n*})$, the space of symmetric 2-tensors. We will let H act on V by the element $a \in H$ sending $S \in V = S^2(\mathbb{R}^{n*})$ into

$$(a^{-1})^t S a^{-1}$$

so that the group multiplication in G is

$$(a, S)(b, T) = (aa', S + (a^{-1})^t T a^{-1}).$$

A matrix version.

$$(a,S)(b,T) = (aa', S + (a^{-1})^t T a^{-1}).$$

This time, I will write the "matrix" version of this multiplication a bit differently: I will write the element (a, S) as the $2n \times 2n$ matrix

$$\begin{pmatrix} a & 0 \\ Sa & (a^{-1})^t \end{pmatrix}.$$

Then

$$\begin{pmatrix} a & 0 \\ Sa & (a^{-1})^t \end{pmatrix} \begin{pmatrix} b & 0 \\ Tb & (b^{-1})^t \end{pmatrix} = \begin{pmatrix} ab & 0 \\ Sab + (a^{-1})^t Tb & ((ab)^{-1})^t \end{pmatrix}$$

$$= \begin{pmatrix} ab & 0 \\ Sab + (a^{-1})^t T a^{-1} ab & ((ab)^{-1})^t \end{pmatrix}$$

as required.

Relation to the symplectic group.

Recall that a $2n \times 2n$ matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is symplectic if and only

$$A^t C = C^t A, \quad B^t D = D^t B, \quad \text{and } A^t D - C^t B = I.$$

So we see that all of the matrices

$$\begin{pmatrix} a & 0 \\ Sa & (a^{-1})^t \end{pmatrix}$$

are symplectic. In other words, we have defined an injective homomorphism of G into $Sp(2n)$.

The Lie algebra.

The Lie algebra $\mathfrak{g}$ of G is then injected into $sp(2n)$ by

$$\iota(A, S) = \begin{pmatrix} A & 0 \\ S & -A^t \end{pmatrix}.$$

Let us use the formula for the moment map of a sympectic representation to compute the moment map of G acting on $\mathbb{R}^{2n}$. We obtain

$$\left\langle \Phi(q,p), \begin{pmatrix} A & 0 \\ S & -A^t \end{pmatrix} \right\rangle = \frac{1}{2}\omega\left(\begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} A & 0 \\ S & -A^t \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \right)$$

$$= \frac{1}{2} q \cdot Sq - \frac{1}{2} q \cdot A^t p - \frac{1}{2} p \cdot Aq.$$

The moment map.

$$= \frac{1}{2} q \cdot Sq - \frac{1}{2} q \cdot A^t p - \frac{1}{2} p \cdot Aq.$$

If we think of the dual space of $gl(n)$ as being $\mathbb{R}^n \otimes \mathbb{R}^{n*}$ we can combine the last two terms as

$$-\langle q \otimes p, A \rangle.$$

The first term is just

$$\frac{1}{2} \langle q \otimes q, S \rangle$$

where we are thinking of the symmetric tensor $q \otimes q$ as lying in $S^2(\mathbb{R}^n)$, the dual space of $V = S^2(\mathbb{R}^{n*})$.

Suppose we consider N copies of $\mathbb{R}^{2n}$ with $Sp(2n)$ acting diagonally. Then the moment map for this action (and hence for the induced action of G) is just the sum of the individual moment maps. Hence

$$\Phi(q_1, \ldots, p_N) = -\pi \left(\sum_{j=1}^{N} q_j \otimes p_j \right) \oplus \frac{1}{2} \sum_j q_j \otimes q_j \tag{6}$$

under the direct sum decomposition

$$\mathfrak{g}^* = \mathfrak{h}^* \oplus S^2(\mathbb{R}^n)$$

and where $q_j \in \mathbb{R}^n, \ p_j \in \mathbb{R}^{n*}$.

The case of *H=Sl(n)*.

In general, a symmetric tensor is is classified into its rank and signature under the action of the full general linear group. So, for example, any positive definite symmetric tensor can be brought to the identity matrix I by the action of a general invertible linear transformation. Under the action of $Sl(n)$ the determinant is preserved, so the most general positive definite symmetric tensor can be brought to cI where $c > 0$. The subgroup of $Sl(n)$ fixing cI is $SO(n)$. So the dimension of this orbit is

$$n^2 - 1 - \frac{n(n-1)}{2} = \frac{n^2 + n}{2} - 1.$$

For example, if $n = 3$ this orbit will be five dimensional. In this case the full group is 14 =8+6 dimensional and the co-adjoint orbit sitting over this V-orbit will either be 10 dimensional or 12 dimensional. Indeed

For example, if $n = 3$ this orbit will be five dimensional. In this case the full group is 14 =8+6 dimensional and the co-adjoint orbit sitting over this V-orbit will either be 10 dimensional or 12 dimensional. Indeed

$$\langle I \odot S, A \rangle = \langle S, A + A^t \rangle = \mathrm{tr}\, s \cdot (A + A^t).$$

So the space of such linear functions is 5=8-3 dimensional. Applying $cI \odot S$ we can move the $\mathfrak{h}^*$ component into an element of $o(3)^*$. If this is the zero element we get a 10 dimensional orbit and if it is non-zero we get a 12 dimensional orbit.

The liquid drop.

We may think of cI as describing a sphere of radius c and hence of volume c^3. We may then think of any element on the five dimensional orbit as describing an ellipsoid with the same volume c^3. So the parameter c^3 represents the volume of a (quadratic approximation to) a "liquid drop".

Notice however that the typical co-adjoint orbit will be 12 dimensional coming from a non-trivial two dimensional orbit of $SO(3)$ acting on $o(3)^*$. It corresponds to a liquid drop with an intrinsic "spin", or as some physicists call it a "vortex motion".