Symplectic geometry
Lecture 14

The convexity theorem.
A normal form for the moment map.
The Duistermaat-Heckman theorem.

Recall: the local cone.

Theorem 4 *Let (M, ω, Φ_M) be a Hamiltonian $\mathbf{T}$ space where $\mathbf{T}$ is a torus. Then every $x \in M$ has a neighborhood O such that there is an open set O' in $\mathfrak{t}^*$ and a cone $C(x)$ in $\mathfrak{t}^*$ such that*

$$\Phi(O) = O' \cap C(x).$$

More precisely, the cone $C(x)$ is

$$C(x) := \Phi_m(x) + \left\{ \mu \in \mathfrak{t}^* \,\middle|\, \mu_{|\mathfrak{t}_x} = \sum r_i \beta_i, \ r_i \geq 0 \right\}$$

where the β_i are the weights of the representation of $\mathbf{T}_x^0$ on $T_x M$.

Let us now assume that M is compact and connected.

We want to convert this local theorem into a global theorem which says that if M is compact, and connected, the image of the moment map is a convex polytope. For this we will use Morse theory.

Critical sets and Hessians.

Let M be a manifold and f a smooth function on M. The **critical set** of f, denoted by $\mathcal{C}_f$ or simply by

$$\mathcal{C}$$

if f is fixed is the set of points at which $df = 0$. At such a point c, for any vector fields X and Y we have

$$D_X(D_Y f)(c) = D_Y(D_X f)(c)$$

and this value depends only on the values $X(c)$ and $Y(c)$. So we get a symmetric bilinear form $d^2 f(c)$ called the **Hessian** of f at c. In local coordinates the Hessian is given by the matrix of second partial derivatives at c.

Morse functions and Morse-Bott functions.

f is called a **Morse function** if its critical set is discrete and the Hessian is non-degenerate at all critical points.

f is called **Morse-Bott** if the connected components $\mathcal{C}^j$ of its critical set $\mathcal{C}$ are smooth submanifolds, and if at each $c \in \mathcal{C}^j$ the kernel of the Hessian at c is the tangent space $T_c\mathcal{C}^j$ to $\mathcal{C}^j$ at c.

In what follows we assume that *M* is compact and connected.

The gradient flow.

If we put a Riemann metric on M we get an identification of T^*M with TM so that df goes over into a vector field ∇f called the **gradient** of f. The critical set $\mathcal{C}$ is the set where this vector field vanishes.

At each $c \in \mathcal{C}^j$ we have an identification of the Hessian $d^2 f$ with a self-adjoint operator $\nabla^2 f$ on $T_c\mathcal{C}^j$ and hence a decomposition

$$T_c M = T_c\mathcal{C} \oplus E_c^+ \oplus E_c^-$$

where E_c^+ is spanned by the eigenvectors corresponding to the positive eigenvalues of $\nabla^2 f(c)$ and E_c^- is spanned by the eigenvectors corresponding to the negative eigenvalues .

Stable and unstable manifolds.

Consider the flow ϕ_t generated by $-\nabla F$. Then (if say M is compact) every point of M tends under this flow to a point of $\mathcal{C}$ as $t \to \infty$. In fact, if $W^s(\mathcal{C}^j)$ denotes the set of points $m \in M$ such that

$$\phi_t(y) \to \mathcal{C}^j \quad \text{as } t \to \infty$$

then $W^s(\mathcal{C}^j)$ is a submanifold called the **stable manifold** of $\mathcal{C}^j$ and is tangent to

$$T\mathcal{C}^j \oplus E^+$$

along $\mathcal{C}$.

Similarly, one defines the unstable manifold $W^-(\mathcal{C}^j)$ as the set of points which tend under ϕ_t to $\mathcal{C}^j$ as $t \to -\infty$. It is tangent to $T\mathcal{C}^j \oplus E^-$ along $\mathcal{C}^j$.

The decompositions into stable or unstable manifolds.

Since every point of M tends to $\mathcal{C}$ as $t \to \pm\infty$ we have the (finite) decompositions

$$M = \bigcup_j W^s(\mathcal{C}^j)$$

and

$$M = \bigcup_j W^u(\mathcal{C}^j).$$

The index.

The **index** n_-^j of $\mathcal{C}^j$ is defined as

$$n_-^j = \dim W^u(\mathcal{C}^j) - \dim \mathcal{C}^j = \mathrm{codim} W^s(\mathcal{C}^j)$$

and is the dimension of the negative eigenspace of the Hessian at any point of $\mathcal{C}^j$.

Similarly define

$$n_+^j := \mathrm{codim} W^u(\mathcal{C}^j).$$

Uniqeness of the minimum.

Proposition 1 *If none of the indices n_-^j is equal to one, then there is a unique j for which $n_-^j = 0$. In other words, there is a unique critical set $\mathcal{C}^j$ along which f takes a local minimum.*

Proof. The condition $n_-^j \neq 1$ means that all the stable manifolds of positive index have codimension at least two, so the complement of their union is connected. Hence there is exactly one stable manifold of index zero. $\square$

Connectedness of the level sets.

Proposition 2 *If none of the indices n^j_- or n^j_+ is equal to one, then for every regular value r of f, the submanifold $f^{-1}(r)$ is connected.*

Proof. We now know that there is a unique $\mathcal{C}^j$ corresponding to a local mimimum of f and a unique $\mathcal{C}^k$ corresponding to a local maximum. Let M_* be the open, dense, connected subset of M obtained by removing all the stable and unstable manifolds corresponding to other stable and unstable manifolds. We know that M_* is connected since all of the manifolds we are throwing away have codimension at least two.

If r is not equal to any of the critical values and

$$\min(f) < r < \max(f)$$

then every trajectory intersects $f^{-1}(r)$ at a unique point. Since r is a regular value $f^{-1}(r)$ is a submanifold of M_* and the map

$$f^{-1}(r) \times \mathbb{R} \to M_*, \quad (m,t) \mapsto \phi_t(m)$$

is a diffeomorphism. This shows that $f^{-1}(r) \cap M_*$ is connected. To prove the proposition it suffices to show that $f^{-1}(r) \cap M_*$ is dense in $f^{-1}(r)$. So let $m \in f^{-1}(r)$ and let U be a connected open neighborhood of m. Since $r \neq \min(f)$ and $r \neq \max(f)$ the set $U \cap M_*$ has non-empty intersection with the set where $f < r$ and with the set where $f > r$. Since $U \cap M_*$ is connected it must have non-empty intersection with $f^{-1}(r)$. $\square$

Connectedness of the level sets, 2.

Proposition 3 *If none of the indices n_-^j or n_+^j is equal to one, then for every value r of f, the submanifold $f^{-1}(r)$ is connected.*

Proof. We know the result for regular values. We must prove it for non-regular values. Let r_j be such a critical value and choose $r > r_j$ so that all values in $(r_j, r]$ are regular. Then there is a continuous surjection of $f^{-1}(r)$ onto $f^{-1}(r_j)$ obtained by following the flow ϕ_t down until it hits $f^{-1}(r_j)$. More precisely, if $f(x) = r$ and $f(\phi_t(x)) > r_j$ for all $t > 0$ sent x to $\lim_{t\to\infty} \phi_t(x)$. Otherwise choose t such that $\phi_t(x) = r_j$ and send x to $\phi_t(x)$. Since $f^{-1}(r)$ is connected it follows that $f^{-1}(r_j)$ is connected. $\square$

Moment maps and Morse-Bott.

Theorem 3 *Let (M, ω, Φ) be a Hamiltonian G-space for a compact Lie group G, and let $A \in \mathfrak{g}$. Then $\langle \Phi, A \rangle$ is a Morse-Bott function whose critical manifolds are all symplectic submanifolds and all its indices are even. In particular, $\langle \Phi, A \rangle$ takes on a unique local minimum value.*

Proof. Let $H \subset G$ be the closure in G of the one parameter subgroup generated by A. So H is a torus. If m belongs to the critical set of $\langle \Phi, A \rangle$, then $A_M(m) = 0$. So m belongs to this critical set if and only if it is a fixed point of H, and we know that the connected components of the fixed point set are symplectic submanifolds.

Suppose that c is such a fixed point set and let $V =$ $T_c M$. By Darboux, we know that in a neighborhood of c there is an H equivariant symplectomprphism of this neighborhood with a neighborhood of 0 in V under the linear action of H on V. So the moment map Φ differs from the moment map Ψ for this linear action by a constant. We can put an H invariant compatible complex structure on V giving rise to weights β_j on $\mathfrak{h}$, the Lie algebra of H and then the moment map Ψ is given by

$$\Psi(\mathbf{z}) = \frac{1}{2} \sum_j \|z_j\|^2 \beta_j$$

once we identify V with $\mathbb{C}^n$. Then

$$\langle \Psi, A \rangle = \frac{1}{2}(p_j^2 + q_j^2)\beta_j(A)$$

where $|z_j|^2 = p_j^2 + q_j^2$. This clearly has even index. $\square$

Recall: the local cone in general.

Now let x be any point of M, not necessarily a fixed point. It will have an isotropy subgroup $\mathbf{T}_x \subset \mathbf{T}$ and the connected component $\mathbf{T}_x^0$ of the identity in $\mathbf{T}_x$ will be a torus. The Lie algebra $\mathfrak{t}_x$ of this torus is a subalgebra of the Lie algebra $\mathfrak{t}$ of $\mathbf{T}$. We let

$$\iota_x : \mathfrak{t}_x \to \mathfrak{t}$$

denote the injection

Theorem 4 *Let (M, ω, Φ_M) be a Hamiltonian $\mathbf{T}$ space where $\mathbf{T}$ is a torus. Then every $x \in M$ has a neighborhood O such that there is an open set O' in $\mathfrak{t}^*$ and a cone $C(x)$ in $\mathfrak{t}^*$ such that*

$$\Phi(O) = O' \cap C(x).$$

More precisely, the cone $C(x)$ is

$$C(x) := \Phi_m(x) + \left\{ \mu \in \mathfrak{t}^* \middle| \mu_{|\mathfrak{t}_x} = \sum r_i \beta_i, \ r_i \geq 0 \right\}$$

where the β_i are the weights of the representation of $\mathbf{T}_x^0$ on $T_x M$.

Let us now assume that M is compact and connected.

Using Morse-Bott.

Proposition 4 *For every $x \in M$ we have $\Phi_M(M) \subset C(x)$.*

Proof. Every closed convex set is the intersection of a family of half-spaces. So there will be a collection $\mathcal{A} = \{A\}$ of elements in $\mathfrak{t}_x$ such that $C(x) - \Phi_M(x)$ consists of the set of all points in $\mathfrak{t}^*$ such that

$$\langle \mu, A \rangle \geq 0, \quad \forall\ A \in \mathcal{A}.$$

The theorem implies that the functions

$$y \mapsto \langle \Phi_M(y) - \Phi_M(x), A \rangle \quad A \in \mathcal{A}$$

have a local mimimum. Since these functions are Morse-Bott, this local minimum must coincide with the global minimum, which is the assertion that $\Phi_M(M) \subset C(x)$. $\square$

The polytope.

Each $x \in M$ has an open neighborhood O_x as in the theorem. Since M is compact, we can choose finitely many of these which cover M and so

$$\Phi_M(M) \subset \bigcap_j C(x_j)$$

where the intersection is over finitely many cones. This intersection is thus a finite polytope Δ. I want to prove that $\Phi_M(M) = \bigcap_j C(x_j)$. Suppose not. Let $\mu \in \Delta$ but $\mu \notin \Phi_M(M)$. Put some Euclidean metric on $\mathfrak{t}^*$ and let $\nu \in \Phi_M(M)$ be a closest point to μ. Such a point exists since $\Phi_M(M)$ is compact. We know that $\nu = \Phi_m(x)$ for some $x \in M$ and $x \in O_{x_j}$ for some j. So both μ and ν lie in $C(x_j)$. Hence so does the entire segment joining them. Since $\nu \in \Phi_M(O_{x_j})$ so are all points in some open interval about ν on this segment, contradicting the hypothesis that ν is the closest point to μ. So there is no such μ.

The convexity theorem.

Let μ be a vertex of the polytope Δ. So $\mu = \Phi_M(x)$ where $d(\Phi_M)_x = 0$. So $\ker d(\Phi_M)_x = T_m M$, and since $\mathfrak{t}_M(m) = d(\Phi_M)_x^{\perp}$ we see that $\mathfrak{t}_M(m) = \{0\}$. We have proved

Theorem 5 [Atiyah-Guillemin-S.]. *Let (M, ω, Φ_M) be a Hamiltonian $\mathbf{T}$ space where $\mathbf{T}$ is a torus and where M is compact and connected. Then $\Phi_M(M)$ is a convex polytope whose vertices are images of fixed points of $\mathbf{T}$ acting on M.*

The coisotropic embedding theorem.

Theorem 4 *Let τ be a closed 2-form of constant rank on a differentiable manifold Z. Then there exists a symplectic manifold (M, ω) and an embedding $\iota : Z \to M$ such that*

- *$\iota(Z)$ is a coisotropic submanifold of M and*
- *$\iota^*\omega = \tau$.*

If (M_1, ω_1, ι_1) and (M_2, ω_2, ι_2) are two such coisotropic embeddings then there exist neighborhoods U_1 of $\iota_1(Z)$ in M_1 and U_2 of $\iota_2(Z)$ in M_2 and a symplectomorphism $f : U_1 \to U_2$ such that

$$\iota_2 = f \circ \iota_1.$$

All the above can be made consistent with the actions of a compact group.

I will only need the second half of the theorem (the uniqueness). So I will only prove this part. The proof of the first part is a straightforward application of the use of homotopy operators. See G and S pp. 316-317.

From (M_1, ω_1, ι_1) we can construct the geometric normal bundle $N_1 = (TM_1)_{|Z}/d\iota_1(TZ)$. This bundle can be identifies with N, the dual bundle to the null foliation of Z. Similarly for (M_2, ω_2, ι_2). So we have two vector bundles N_1 and N_2 over Z and a vector bundle isomorphism

$$A : N_1 \to N_2$$

between them.

Using exponential maps we can identify neighborhoods of the zero sections with tubular neighborhoods of the images and regard A as a diffeomorphism from a neighborhood U_1 of $\iota_1(Z)$ in M_1 to a neighborhood U_2 of $\iota_2(Z)$ in M_2. By construction, the restriction of ω_1 to Z regarded as the zero section of N_1 identifies the fiber of N_1 with the dual space of the null foliation of Z and similarly for N_2 which means that

$$(A^*\omega_2)_{|Z} = (\omega_1)_{|Z}.$$

We can now apply one of our Weinstein theorems to find a g such that

$$g^*(A^*\omega_2)_{|Z}(\omega_1)$$

in some neighborhood of Z so $f := A \circ g$ does the job. $\square$

Isotropic orbits.

Let M be a Hamiltonian G-manifold with moment map $\Phi : M \to \mathfrak{g}^*$.

We know that

$$\ker d\Phi_p = (T(G \cdot p)_p)^{\perp} \qquad (1.7)$$

where the right hand side denotes the orthogonal complement relative to the symplectic form at p of the tangent space to the G-orbit through p.

It follows that

Proposition 1.6.1 *Suppose that $X = G \cdot p$ is a G-orbit such that Φ is constant on X. Then X is an isotropic submanifold of M.*

Particular cases of interest are:

- G is commutative, so the coadjoint action is trivial, and elements of $\mathbf{g}^*$ are fixed points
- or, for general G, where the orbit lies in $\Phi^{-1}(0)$.

Variation of the reduced spaces.

Let (M, ω, Φ_M) be a Hamiltonian G-space where G is a compact Lie group. If 0 is a regular value of the moment map, so is μ if μ lies in a small enough neighborhood of 0. We let M_μ denote the Marsden-Weinstein reduced space at μ. We want to investigate the relation between M_μ and M_0 for μ sufficiently close to 0. For this we will use a normal form for the moment map for points near

$$Z := \Phi^{-1}(0).$$

As usual, with will let $\iota : Z \to M$ be the inclusion as a submanifold and let $\pi : Z \to M_0$ be the projection.

The one form θ .

We can construct a one form $\theta \in \Omega^1(Z, \mathfrak{g})$ that satisfies the conditions

$$g^*\theta = \mathrm{Ad}_g^*\theta, \ \forall\ g \in G, \quad \iota(A_Z)\theta = -A \ \ \forall\ A \in \mathfrak{g}$$

as follows:

The fact that 0 is a regular value of Φ means that the evaluation map $A \mapsto A_Z(p)$ is injective at all $p \in Z$. In other words, we have an injection

$$Z \times \mathfrak{g} \to TZ \quad (m, A) \mapsto (m, -A_Z(m))$$

as a sub-bundle. Put a G invariant Riemann metric on Z, and let $\rho : TZ \to Z \times \mathfrak{g}$ be orthogonal projection. Then take $\theta : TZ \to \mathfrak{g}$ to be the $\mathfrak{g}$ valued one form that forgets about the Z component in $Z \times \mathfrak{g}$.

A non-degenerate two form.

Let $\mathrm{pr}_1 : Z \times \mathfrak{g}^* \to Z$ and $\mathrm{pr}_2 : Z \times \mathfrak{g}^* \to \mathfrak{g}^*$ be projections onto the first and second factor. Consider the two form

$$\mathrm{pr}_1^* \pi^* \omega_0 + \langle d\,\mathrm{pr}_2, \wedge\theta\rangle$$

where ω_0 is the symplectic form on M_0. I claim that this two form is non-degenerate. Indeed, let (v, w) be a tangent vector to $Z \times \mathfrak{g}^*$ at (m, ξ) so that $v \in T_m Z$ and $w \in \mathfrak{g}^*$ under the identification of $T_\xi \mathfrak{g}^*$ with $\mathfrak{g}^*$. Under this identification, $d(\mathrm{pr}_2)_{(m,\xi)} w = w$. So

$$i(v, w)\,(\mathrm{pr}_1^* \pi^* \omega_0 + \langle d\,\mathrm{pr}_2, \wedge\theta\rangle) = \pi^*(i(d\pi_m(v))\omega_0) - \langle d\,\mathrm{pr}_2, \theta(v)\rangle + \langle w, \theta\rangle.$$

$$i(v,w)\,(\mathrm{pr}_1^*\,\pi^*\omega_0 + \langle d\,\mathrm{pr}_2, \wedge\theta\rangle) = \pi^*(i(d\pi_m(v))\omega_0) - \langle d\,\mathrm{pr}_2, \theta(v)\rangle + \langle w, \theta\rangle.$$

This can not vanish unless $\theta(v) = 0$ since the first and third terms are forms on Z which can not cancel a form on $\mathfrak{g}^*$. The first term vanishes on all tangents to the orbits. But then the last term can not vanish on all tangents to the orbits (i.e. elements in the image of $\mathfrak{g}$) unless $w = 0$. The map $d\pi_m$ is injective on those v which satisfy $\theta(v) = 0$ and since ω_0 is symplectic, we conclude that $v = 0$ if the above interior product vanishes.

The model.

Now consider the two form

$$\sigma := \mathrm{pr}_1^* \pi^* \omega_0 + d\langle \mathrm{pr}_2, \theta \rangle.$$

This form is closed, and along the zero section $Z \times \{0\}$ coincides with the two form constructed above. So σ is a symplectic form in some neighborhood U of the zero section. The one form

$$\langle \mathrm{pr}_2, \theta \rangle$$

is G invariant and so $D_{A_U} \langle \mathrm{pr}_2, \theta \rangle = 0$ and so

$$i(A_U) d\langle \mathrm{pr}_2, \theta \rangle + d(i(A_U) \langle \mathrm{pr}_2, \theta \rangle = 0$$

or

$$i(A_U) d\langle \mathrm{pr}_2, \theta \rangle = d\langle \mathrm{pr}_2, A \rangle$$

while $i(A_U)\,\mathrm{pr}_1^* \pi^* \omega_0 = 0$. In other words, the action of G on U is Hamiltonian with moment map

$$\mathrm{pr}_2 : U \to \mathfrak{g}^*.$$

Using the co-isotropic embedding theorem.

So if we consider Z as the zero section of $U \subset Z \times \mathfrak{g}^*$, then Z is the inverse image of 0 under the moment map pr_2 and so is a co-isotropic submanifold of U. By the co-isotropic embedding theorem we know that there is a symplectomorphism of some neighborhood of Z in M onto some neighborhood of Z in U which restricts to the identity on Z. In the proof of the co-isotropic embedding theorem we made some choices (or Riemann metric etc.) which we could take to be to be G invariant, and then our symplectomorphism would be G equivariant. The moment maps Φ_M and $\Phi_U = \mathrm{pr}_2$ both take the value 0 on Z. Hence our symplectomorpism carries one moment map into the other. To summarize:

The normal form.

Theorem 5 *Let* (M, ω, Φ_M) *be a Hamiltonian* G*-space where* G *is a compact Lie group. Let* Z *be the zero level set of the moment map* $\Phi_M : M \to \mathfrak{g}^*$ *where* 0 *is assumed to be a regular value of* Φ_M*. Let* $M_0 = Z/G$ *be the reduced space (assumed to be a manifold) and let* ω_0 *be the reduced symplectic form on* M_0*. We can choose a* $\mathfrak{g}$ *valued one form* θ *on* Z *such that*

$$g^*\theta = \mathrm{Ad}_g^*\,\theta,\ \forall\ g \in G, \quad \iota(A_Z)\theta = -A \ \ \forall\ A \in \mathfrak{g}.$$

Let $\mathrm{pr}_1 : Z \times \mathfrak{g}^* \to Z$ *and* $\mathrm{pr}_2 : Z \times \mathfrak{g}^* \to \mathfrak{g}^*$ *be projections onto the first and second factor. Then*

$$\sigma := \mathrm{pr}_1^* \pi^* \omega_0 + d\langle \mathrm{pr}_2, \theta \rangle$$

is symplectic in some neighborhood U *of* $Z \times \{0\}$ *in* $Z \times \mathfrak{g}^*$ *and* G *acts in Hamiltonian fashion on* U *with moment map*

$$\Phi_U = \mathrm{pr}_2 \,.$$

There is a G*-equivariant symplectomorphism of some neighborhood of* Z *in* M *with some neighborhood of* Z *(identified with* $Z \times \{0\}$*) in* $Z \times \mathfrak{g}^*$ *intertwining* Φ_M *and* pr_2.

Nearby reduced spaces.

Now consider the Marsden-Weinstein reduced space M_μ for a μ near 0 in our model description. We must take the inverse image of 0 in $\mathfrak{g}^*$ of $\mathcal{O}^- \times U$ under pr_2 and then divide by G where $\mathcal{O}$ is the G orbit through μ. Since $\mathcal{O}^-$ is symplectomorphic to $G\cdot(-\mu)$ we conclude that the Marsden-Weinstein reduced M_μ is symplectomorphic to

$$(Z \times G \cdot (-\mu))/G \tag{1}$$

and that the pull back of the symplectic form on M_μ to $Z \times G \cdot (-\mu)$ is given by

$$\pi^*\omega_0 + d\langle \iota, \theta \rangle \tag{2}$$

where $\iota : G \cdot \mu \to \mathfrak{g}^*$ is the injection of $\mathcal{O} = G \cdot \mu$ into $\mathfrak{g}^*$ as a co-adjoint orbit. Notice that the space M_μ is fibered over M_0.

The case of a torus.

Let us now specialize to the case where $G = \mathbb{T}^n$ is a torus. Then the adjoint and the co-adjoint actions are trivial, so M_μ is diffeomorphic to M_0 for μ near 0. What is changing is the symplectic structure. In other words, if we identify M_μ with M_0 as a differentiable manifold, we must describe how the symplectic structure on M_0 is changing with μ.

Notice that since the adjoint representation is trivial, the form θ of the theorem is invariant. So for any $A \in \mathfrak{g}$ we have

$$D_{A_U}\theta = 0.$$

Since $i(A_U)\theta = -A$ is a constant, we have

$$D_{A_U}\theta = i(A_U)d\theta.$$

So the $\mathfrak{g}$ valued two form $d\theta$ on Z satisfies

$$i(A_U)d\theta = 0 \quad \text{and} \quad D_{A_U}d\theta = 0 \quad \forall A \in \mathfrak{g}.$$

The two form is basic.

So the $\mathfrak{g}$ valued two form $d\theta$ on Z satisfies

$$i(A_U)d\theta = 0 \text{ and } D_{A_U}d\theta = 0 \quad \forall A \in \mathfrak{g}.$$

In other words, it is basic relative to the fibration of Z over M_0, so that there is a closed two form F on M_0 so that $\pi^*F = d\theta$. Since the ι in (2) is a constant, we can write the closed two form in (2) as

$$\pi^*\left(\omega_0 + \langle \mu, F \rangle\right).$$

So we have proved

The Duistermaat-Heckman theorem.

Theorem 2 [Duistermaat-Heckman]. *Let 0 be a regular value of the moment map for a torus action reduced manifold (M_0, ω_0). Then all the nearby reduced spaces M_μ are diffeomorphic to M_0. The variation of the symplectic structure can be described as follows: Then there is a closed $\mathfrak{g}$ valued two form F on M_0 such that the symplectic form ω_μ is given by*

$$\omega_\mu = \omega_0 + \langle \mu, F \rangle. \tag{3}$$

Dependence on choices.

The form F depended on some choices. Once we have introduced the concepts of connections on a principal bundle and their curvatures, we will see that the cohomology class of F is independent of all choices. Thus the cohomology class $[\omega_\mu]$ of ω_μ is independent of all choices.

The volume of the reduced space.

The form ω_μ in (3) depends linearly on μ. This means that the Liouville form is a polynomial in μ of degree at most $\frac{1}{2}\dim M_0 = \frac{1}{2}(\dim M - \dim G)$. Integrating over M_0 then shows that the symplectic volume of M_μ is a polynomial in μ of at most the same degree.