Symplectic Geometry
Lecture 13

The moment map and reduction.

The cotangent bundle of a group.

A normal form for the moment map near an isotropic orbit.

Review: symplectic actions of a group.

Let (M, ω) be a symplectic manifold. A G-action $g \mapsto \mathcal{A}_g$ on M is called **symplectic** if $\mathcal{A}_g \in \mathrm{Symp}(M)$ for all $g \in G$. In other words, if

$$\mathcal{A}_g^* \omega = \omega \quad \forall\, g \in G.$$

Similarly, an action of a Lie algebra $\mathfrak{g}$ is called symplectic if $A_M \in \mathfrak{X}(M, \omega)$ for all $A \in \mathfrak{g}$. Clearly, if $\mathfrak{g}$ is the Lie algebra of G, then the $\mathfrak{g}$ action defined by a symplectic G action is symplectic.

Review: Weakly Hamiltonian actions

A symplectic G-action or a $\mathfrak{g}$-action is called **weakly Hamiltonian** if all the vector fields A_M are Hamiltonian. In other words, if for each $A \in \mathfrak{g}$ there is a smooth function $\Phi(A)$ on M such that

$$A_M = X_{\Phi(A)}.$$

One can always choose $\Phi(A)$ to depend linearly on A, by fixing the values of $\Phi(A_i)$ for the A_i in a basis of $\mathfrak{g}$ and then extending linearly. Then the map

$$A \mapsto \Phi(A)$$

can be viewed as a $\mathfrak{g}^*$ valued function on M:

$$\Phi \in C^\infty(M) \otimes \mathfrak{g}^*.$$

Review: Moment maps.

In other words, a symplectic G-action on a symplectic manifold (M, ω) is weakly Hamiltonian if there is a smooth map, called the **moment map**

$$\Phi : M \to \mathfrak{g}^*$$

i.e. $\Phi \in C^\infty(M) \otimes \mathfrak{g}^*$ such that

$$i(A_M)\omega = d\langle \Phi, A \rangle \quad \forall\ A \in \mathfrak{g}. \tag{4}$$

Review: Hamiltonian actions.

Definition 4 *A weakly Hamiltonian G-action is called* **Hamiltonian** *with moment map* Φ *if the moment map can be (and has been) chosen so as to be an equivariant map from* M *to* $\mathfrak{g}^*$ *relative to the co-adjoint action of* G *on* $\mathfrak{g}^*$.

Similarly one defines moment maps for Hamiltonian $\mathfrak{g}$ actions: one requires Φ to be $\mathfrak{g}$ equivariant in this case.

Review: The derivative of the moment map.

Let us be given a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω) with moment map $\Phi : M \to \mathfrak{g}^*$. The defining property of the moment map is

$$d\langle \Phi, A\rangle = i(A_M)\omega \quad \forall\ A \in \mathfrak{g}.$$

In this equation, A is a constant (as a function on M) so we can rewrite this as

$$\langle d\Phi_m(v), A\rangle = \omega_m(A_M(m), v) \quad v \in T_mM, \ \ A \in \mathfrak{g}. \qquad (4)$$

Review: The evaluation map and its transpose.

$$\langle d\Phi_m(v), A\rangle = \omega_m(A_M(m), v) \quad v \in T_mM, \quad A \in \mathfrak{g}. \qquad (4)$$

Here is a way of thinking about this central equation: The action of G on M gives a linear map

$$\mathbf{ev}_M(m) : \mathfrak{g} \to T_mM, \qquad A \mapsto A_M(m)$$

for each $m \in M$. Let us call this map the **evaluation** map. The transpose of the evaluation map would be a linear map from $T_m^*M \to \mathfrak{g}^*$. The symplectic form ω_m gives us an isomorphism

$$T_mM \to T_m^*M, \quad v \mapsto \omega_m(\cdot, v).$$

Review: The moment map and the evaluation map.

$$\langle d\Phi_m(v), A\rangle = \omega_m(A_M(m), v) \quad v \in T_m M, \quad A \in \mathfrak{g}. \qquad (4)$$

$$T_m M \to T_m^* M, \quad v \mapsto \omega_m(\cdot, v).$$

Using this isomorphism, we can regard the transpose as a map $T_m M \to \mathfrak{g}^*$ and (4) says that

Proposition 1 $d\Phi_m : T_m M \to \mathfrak{g}^*$ *is the transpose of the evaluation map* $\mathfrak{g} \to T_m M$ *when we identify* $T_m^* M$ *with* $T_m M$ *using* ω_m.

Review: the kernel of the derivative of the moment map.

Let $\mathfrak{g}_M(m)$ denote the subspace of $T_m M$ consisting of all the $A_M(m),\ A \in \mathfrak{g}$. So $\mathfrak{g}_M(m)$ is the image of the evaluation map at m. Geometrically, it is the tangent space to the orbit $G \cdot m$ at m. Since the kernel of the transpose of a linear map is the annihilator space of the image, we conclude that

$$\ker d\Phi_m = \mathfrak{g}_M(m)^\perp \tag{5}$$

where $\perp$ means the perpendicular relative to ω_m.

Review: A transitive Hamiltonian space covers a coadjoint orbit.

For example, suppose that G acts transitively on M so that $\mathfrak{g}_M(m) = T_m M$ at all points. Then the kernel of $d\Phi_m$ is $\{0\}$ at all m. In other words, Φ is an immersion. Since Φ is equivariant and G acts transitively on M, the image of Φ must be a single orbit $\mathcal{O}$ of the co-adjoint action of G on $\mathfrak{g}^*$:

$$\Phi(M) = \mathcal{O}, \quad \mathcal{O} = G \cdot \Phi(m), \quad m \in M.$$

So the map

$$\Phi : M \to \mathcal{O}$$

is a covering map.

Review: The Kostant-Souriau theorem.

Putting it all together we obtain:

Theorem 2 [Kostant-Souriau]. *Any co-adjoint orbit $\mathcal{O}$ carries a unique symplectic form σ for which the injection*

$$\iota : \mathcal{O} \to \mathfrak{g}^*$$

is the moment map. At each $\mu \in \mathcal{O}$ this symplectic form is given by

$$\sigma_\mu(A_\mathcal{O}(\mu), B_\mathcal{O}(\mu)) = \langle \mu, [A, B] \rangle.$$

If M is any symplectic manifold G acts in a Hamiltonian fashion and the action is transitive, then the moment map $\Phi : M \to \mathfrak{g}^*$ is in fact a covering map of some orbit $\mathcal{O}$ of G acting on $\mathfrak{g}^*$ and the symplectic form on G is the pull-back via Φ of the symplectic form σ on $\mathcal{O}$.

Review: the image of the derivative of the moment map.

$d\Phi_m$ maps T_mM to $T_{\Phi(m)}(\mathfrak{g}^*)$. Since $\mathfrak{g}^*$ is a vector space, we may identify $T_{\Phi(m)}(\mathfrak{g}^*)$ with $\mathfrak{g}^*$ and hence think of $d\Phi_m$ as a map

$$d\Phi_m : T_mM \to \mathfrak{g}^*.$$

The image of this map will be a subspace of $\mathrm{im}(d\Phi_m) \subset \mathfrak{g}^*$. So the annihilator space $(\mathrm{im}\, d\Phi_m)^0$ of this subspace will be that subspace of $\mathfrak{g}$ consisting of all $A \in \mathfrak{g}$ such that $\langle \mu, A \rangle = 0$ for all $\mu \in \mathrm{im}\, d\Phi_m$. Prop.1 tells us that

$$(\mathrm{im}\, d\Phi_m)^0 = \{A \in \mathfrak{g} | A_M(m) = 0\}. \tag{6}$$

Review: The stabilizer subgroup.

$$(\mathrm{im}\, d\Phi_m)^0 = \{A \in \mathfrak{g} | A_M(m) = 0\}. \tag{6}$$

The right hand side of this equation is the Lie algebra of the subgroup $G_m \subset G$ consisting of those elements which fix m sometimes called the **stabilizer group** of m. As a corollary of this equation we see that

Proposition 2 *$d\Phi_m$ is surjective if and only if the stabilizer subgroup of m is discrete.*

Clean intersection.

Let $f : P \to Q$ be a smooth map between two differentiable manifolds, and let W be an embedded submanifold of Q. We say that f intersects W **cleanly** if the following two conditions hold:

- $f^{-1}(W)$ is a submanifold of P and
- at each $p \in P$, $T_p(f^{-1}(W)) = df_p^{-1}(TW_{f(p)})$.

For example, if f is transversal to W meaning that at each $p \in f^{-1}(W)$

$$df_p(T_pP) + T_{f(p)}W = T_{f(p)}Q$$

then the implicit function theorem guarantees that the two conditions hold. But cleanness is a more general property than transversality.

$\Phi^{-1}(\mathcal{O})$ is co-isotropic.

Let M be a Hamiltonian G-manifold with moment map $\Phi : M \to \mathfrak{g}^*$ and let $\mathcal{O}$ be a G-orbit in $\mathfrak{g}^*$.

The following theorem is a vast generalization of our construction of the symplectic structure on projective space:

Theorem 1 [Kazhdan, Kostant and S, 1978.] *If $\Phi : M \to \mathfrak{g}^*$ intersects $\mathcal{O}$ cleanly, then $\Phi^{-1}(\mathcal{O})$ is coisotropic and the leaf of the null foliation passing through a point $m \in \Phi^{-1}(\mathcal{O})$ is the orbit of m under $G^0_{\Phi(m)}$, the connected component of the isotropy group of $\Phi(m)$ under the action of G on $\mathfrak{g}^*$.*

Proof, I.

Proof. Let

$$Q := \Phi^{-1}(\mathcal{O}) \quad \text{and} \quad m \in Q.$$

The clean intersection hypothesis says that Q is a submanifold of M and that

$$T_m Q = (d\Phi_m)^{-1}(T_{\Phi(m)}\mathcal{O}).$$

The tangent space $T_{\Phi(m)}\mathcal{O}$ to $\mathcal{O}$ at $\Phi(m)$ consists of all vectors of form $A_{\mathfrak{g}^*}(\Phi(m))$ as A ranges over $\mathfrak{g}$. The equivariance of Φ implies that

$$d\Phi_m(A_M(m)) = A_{\mathfrak{g}^*}(\Phi(m)).$$

So

$$T_m Q = \mathfrak{g}_M(m) + \ker d\Phi_m.$$

Proof, 2.

$$T_m Q = g_M(m) + g_M(m)^\perp$$

as a subspace of $T_m M$ where $\perp$ is relative to the symplectic form ω_m. Now

$$(g_M(m) + g_M(m)^\perp)^\perp = g_M(m) \cap g_M(m)^\perp \subset g_M(m) + g_M(m)^\perp.$$

This shows that $T_m Q$ is co-isotropic with null space at m given by

$$(T_m Q)^\perp = g_M(m) \cap g_M(m)^\perp.$$

Now $A_M(m) \in \mathfrak{g}_M(m)^\perp = \ker d\Phi_m$ if and only if $A_{\mathfrak{g}^*}(\Phi(m)) = 0$. So $(T_m Q)^\perp$ consist of all vectors of the form $A_M(m)$ where A is in the Lie algebra of $G_{\Phi(m)}$. $\square$

Relation to the projective space example.

In our example of projective space $G = \mathbb{T}^1$ was the circle acting diagonally on $\mathbb{C}^n$. Since G is commutative, the action of G on $\mathfrak{g}^*$ is trivial, so the coadjoint orbits are points. We found that if we took a non-zero point in $\mathfrak{g}^* = \mathbb{R}$ the inverse image was a sphere which was coisotropic and its foliation (by circles) was a fibration over projective space which then acquired a symplectic structure.

In the general case we should discuss what conditions will guarantee that the null foliation of $Q = \Phi^{-1}(\mathcal{O})$ is a fibration. I may go into this later. But in many cases, just as in the case of projective space, we can check directly that the null foliation of Q is a fibration over a base B which then is a symplectic manifold.

The dimension of the base.

Assuming that the fibration of Q is fibrating, let us compute the dimension of B:

For $m \in Q$ we know that $\ker d\Phi_m = \mathfrak{g}_M(m)^\perp$ and therefore

$$\dim \ker d\Phi_m = \dim M - \dim \mathfrak{g}_M(m) = \dim M - \dim \mathfrak{g} + \dim \mathfrak{i}_m$$

where $\mathfrak{i}_m$ is the Lie algebra of the isotropy group at m, i.e. $\mathfrak{i}_m \subset \mathfrak{g}$ consists of those $A \in \mathfrak{g}$ such that $A_M(m) = 0$. So if $\mathfrak{i}_{\Phi(m)}$ denotes the Lie algebra of the isotropy group of $\Phi(m)$ we have

$$\begin{aligned} \dim Q &= \dim T_m Q \\ &= \dim \mathcal{O} + \dim \ker d\Phi_m \\ &= \dim \mathfrak{g} - \dim \mathfrak{i}_{\Phi(m)} + \dim M - \dim \mathfrak{g} + \dim \mathfrak{i}_m \end{aligned}$$

so

$$\dim Q = \dim M + \dim \mathfrak{i}_m - \dim \mathfrak{i}_{\Phi(m)}. \qquad (1)$$

$$\dim Q = \dim M + \dim \mathfrak{i}_m - \dim \mathfrak{i}_{\Phi(m)}. \tag{1}$$

Since $\dim(T_m Q)^\perp = \dim M - \dim Q$ we see that

$$\dim(T_m Q)^\perp = \dim \mathfrak{i}_{\Phi(m)} - \dim \mathfrak{i}_m. \tag{2}$$

Since $\dim B = \dim Q - \dim(T_m Q)^\perp$ we see that

$$\dim B = \dim M - 2(\dim \mathfrak{i}_{\Phi(m)} - \dim \mathfrak{i}_m). \tag{3}$$

The orbit reduced space.

$$\dim B = \dim M - 2(\dim \mathfrak{i}_{\Phi(m)} - \dim \mathfrak{i}_m). \tag{3}$$

An important special case is where $\mathfrak{i}_{\Phi(m)} = \mathfrak{g}$ (which be the case if G is abelian) and $\mathfrak{i}_m = \{0\}$ which will be the case if the isotropy group of m is discrete. Then (3) becomes

$$\dim B = \dim M - 2\dim \mathfrak{g}. \tag{4}$$

In any event, the space B is called the **orbit reduced space** or, less modestly, the KKS reduced space.

The reduced Hamiltonian.

If $\mathcal{H}$ is a function on M which is invariant under H, then the Hamiltonian flow corresponding to $\mathcal{H}$ preserves Q and its restriction to Q is of the form $\pi^*\mathcal{H}_{red}$ where $\mathcal{H}_{red}$, called the reduced Hamiltonian, is a function on B and $\pi : Q \to B$ is the null fibration. The trajectories of the Hamiltonian flow on Q project down to the trajectories of the Hamiltonian $\mathcal{H}_{red}$ on B.

For example, consider the case of total linear momentum, where $G = \mathbb{R}^3$ acting as "simultaneous translations" on the configuration space of n particles moving in $\mathbb{R}^3$. The phase space M (= cotangent bundle) for this system has dimension $6n$. If the Hamiltonian is invariant under G, the reduced space has dimension $6n - 6$. We have "reduced" the number of variables.

Example: angular momentum in the plane.

Consider the motion of a particle of mass 1 on $\mathbb{R}^2$ in a potential $V = V(q) = V(q_1, q_2)$. It is described by the Hamiltonian H on $T^*\mathbb{R}^2$

$$H(q,p) = \frac{1}{2}(p_1^2 + p_2^2) + V(q).$$

Suppose the potential has rotational symmetry i.e. that it depends only on $r = \|q\|$. Then H is invariant under the cotangent lift of the rotation action of $G = \mathbb{T}^1$. The moment map for this action is angular momentum $\Phi(q,p) = p_2 q_1 - p_1 q_2$. In polar coordinates (r, θ) on $\mathbb{R}^2$ and corresponding cotangent coordinates on $T^*\mathbb{R}^2$

$$H(r, \theta, p_r, p_\theta) = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) + V(r)$$

Example: angular momentum in the plane. Using polar coordinates.

$$H(r,\theta,p_r,p_\theta) = \frac{1}{2}\left(p_r^2 + \frac{1}{r^2}p_\theta^2\right) + V(r)$$

and

$$\Phi(r,\theta) = p_\theta$$

where the symplectic form on $T^*\mathbb{R}^2$ is

$$\omega = dr \wedge dp_r + d\theta \wedge dp_\theta.$$

All the orbits of G acting on $\mathbb{R} = \mathfrak{g}^*$ are points. Since G acts freely on the set where $p_\theta = r^2\dot{\theta} \neq 0$, the differential of Φ is not zero at any of these points. So if $\mu \neq 0$ then $Q = \Phi^{-1}(\mu)$ is the co-isotropic submanifold of $T^*\mathbb{R}^2$ given by $p_\theta = \mu$. Since $dp_\theta = 0$ on Q, we see that the restriction of ω to Q is $dr \wedge dp_r$.

Example: angular momentum in the plane: the effective potential.

So it is clear that the null foliation is given by the circles $r =$ const., $p_r =$ const. and so the base B has coordinates r, p_r and symplectic form $dr \wedge dp_r$. The reduced Hamiltonian is

$$H_{red}(r, p_r) = \frac{1}{2}p_r^2 + \frac{\mu^2}{2r^2} + V(r).$$

In the physics literature this is written as

$$H_{red} = \frac{1}{2}p_r^2 + V_{eff}(r), \quad V_{eff} = \frac{\mu^2}{2r^2} + V(r)$$

and V_{eff} is called the **effective potential**. The equations of motion of B are those of a one particle moving in one dimension with effective potential V_{eff}.

Example: angular momentum in the plane, solving the reduced equations.

$$H_{red} = \frac{1}{2}p_r^2 + V_{eff}(r), \quad V_{eff} = \frac{\mu^2}{2r^2} + V(r)$$

Using conservation of energy for this dynamical system in one dimension we have

$$\frac{1}{2}p_r^2 + V_{eff}(q) = \frac{1}{2}\dot{r}^2 + V_{eff}(q) = E = \text{const.}.$$

In other words,

$$\dot{r} = \sqrt{2(E - V_{eff}(r))}.$$

We can solve this equation for t as a function of r

$$t - t_0 = \int_{r_0}^{r} \frac{dr}{\sqrt{2(E - V_{eff}(r)}}$$

and then invert to get r as a function of t. Using $r^2\dot{\theta} = \mu$ we can then solve for θ as a function of t.

Back to general considerations.

We continue the study of a Hamiltonian G-action on a symplectic manifold (M, ω) whose moment map $\Phi_M : M \to \mathfrak{g}^*$ intersects a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ cleanly with

$$Q = \Phi_M^{-1}(\mathcal{O})$$

so that Q is a coisotropic submanifold of M. We assume that the null foliation of Q is fibrating so that we have a fibration

$$\pi : Q \to B$$

where (B, σ) is a symplectic manifold with

$$\pi^*\sigma = \iota^*\omega,$$

where $\iota : Q \to M$ is the injection of Q as a submanifold of M.

The reduced moment map.

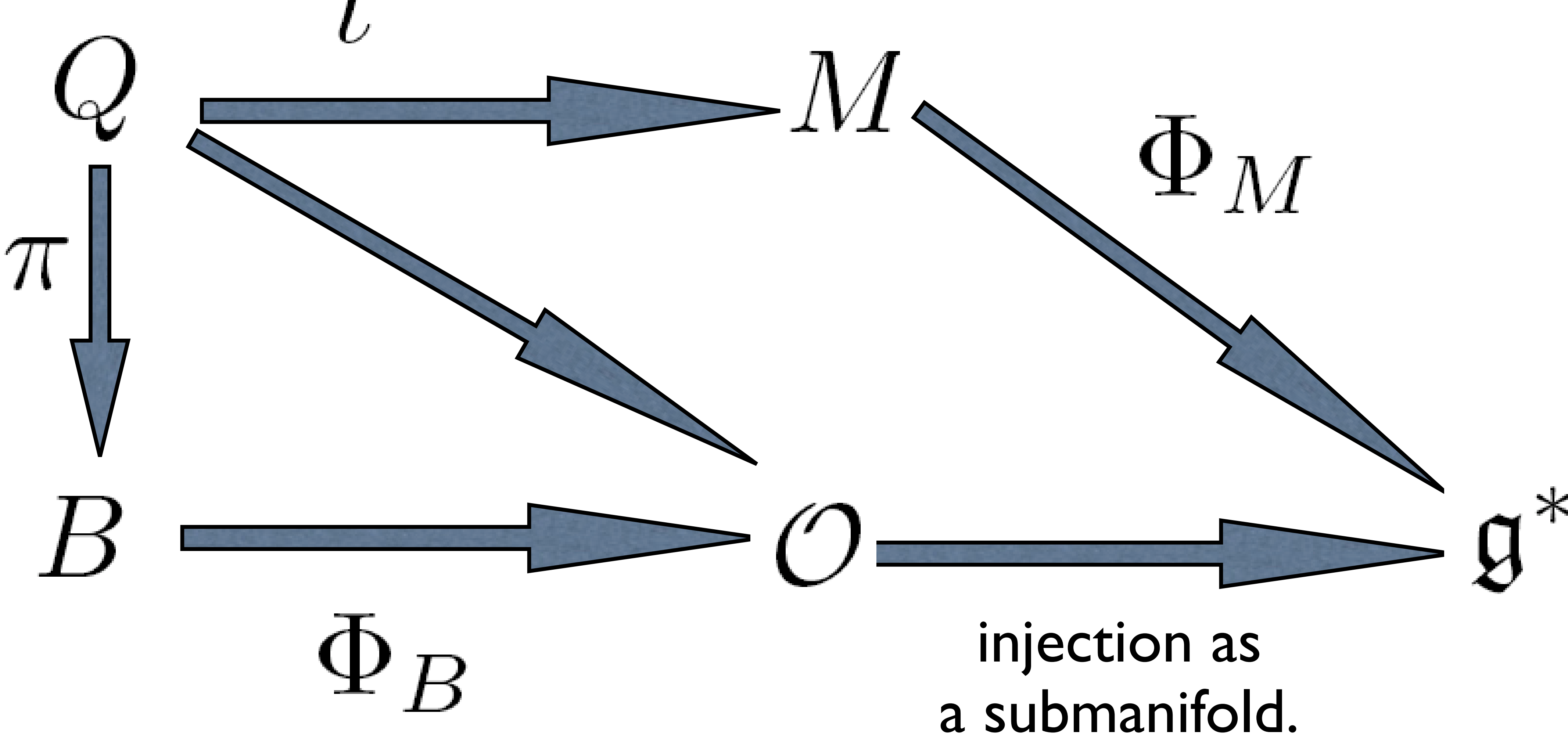
$$\pi : Q \to B \qquad \pi^*\sigma = \iota^*\omega,$$

The group G preserves Q and its null foliation, and hence acts on B so that the map π is G-equivariant. Thus, for every $A \in \mathfrak{g}$ we get vector fields A_Q on Q and A_B on B and they are π-related. So

$$\pi^*\left(i(A_B)\sigma\right) = i(A_Q)\iota^*\omega = \iota^* d\langle \Phi_M, A\rangle.$$

Since the null foliation through $q \in Q$ is the orbit of the connected component of the isotropy group $G_{\Phi_M(q)}$ acting on q, the function Φ_M is constant on the null foliation of Q and hence descends to a function Φ_B which is a moment map for the G-action on B, and Φ_B takes values in the single orbit $\mathcal{O}$.

Diagram.


ι
Q
M
Φ_M
π
B
$\mathcal{O}$
$\mathfrak{g}^*$
Φ_B
injection as
a submanifold.

The Marsden-Weinstein reduced space.

Since $\mathcal{O}/G$ is a point, this suggests that we investigate B/G. We will give an alternative description of B/G which will allow us to conclude that B/G is a manifold, indeed a symplectic manifold, called the **Marsden-Weinstein reduced space**.

Let M and N be Hamiltonian G-spaces with moment maps Φ_M and Φ_N. Then $N^- \times M$ is a Hamiltonian G-space with moment map $\Phi_M - \Phi_N$ (in the obvious notation). In particular, consider $N = \mathcal{O}$ as a Hamiltonian G-space with moment map $\Phi_{\mathcal{O}} = \iota_{\mathcal{O}}$, the injection of $\mathcal{O}$ into $\mathfrak{g}^*$. The moment map

$$\Phi_{\mathcal{O}^- \times M} = \Phi_M - \iota_{\mathcal{O}}$$

has the property that

$$\Phi^{-1}_{\mathcal{O}^- \times M}(0) = \{(m, \mu) | \Phi_M(m) = \mu \ \text{ and } \quad \mu \in \mathcal{O}\}.$$

$$\Phi^{-1}_{\mathcal{O}^-\times M}(0) = \{(m,\mu)|\Phi_M(m) = \mu \ \text{ and } \quad \mu \in \mathcal{O}\}.$$

The hypothesis that Φ_M intersect $\mathcal{O}$ cleanly is equivalent to the hypothesis that $\Phi_{\mathcal{O}^-\times M}$ intersect $\{0\}$ cleanly. So if we make this hypothesis, we see that $\Phi^{-1}_{\mathcal{O}^-\times M}(0)$ is a co-isotropic submanifold of $\mathcal{O}^- \times M$ whose null foliation through any point is the orbit of that point under the connected component of the isotropy group of $\{0\}$. The isotropy group of $\{0\}$ is all of G. So if we assume that G is connected the null foliation through any point $(m, \Phi_M(m))$ of $\Phi^{-1}_{\mathcal{O}^-\times M}(0)$ is the orbit of $(m, \Phi_M(m))$ under G. Since $\Phi_M(m) \in \mathcal{O}$, we know that $m \in Q$, and so $b = \pi(m) \in B$ is such that $\Phi_B(b) = \Phi_M(m)$. So the quotient $\Phi^{-1}_{\mathcal{O}^-\times M}(0)/G$ is the same as $\Phi^{-1}_{\mathcal{O}^-\times B}(0)/G$ which is the same as B/G.

Rewriting the Marsden-Weinstein reduced space.

Another way of writing the Marsden-Weinstein reduced space is to observe that since G acts transitively on $\mathcal{O}$, we can, set theoretically, identify $\Phi^{-1}_{\mathcal{O}^- \times M}(0)/G$ with $\Phi^{-1}_M(\mu)/G_\mu$ for any $\mu \in \mathcal{O}$. This is the way that the Marsden-Weinstein reduced space is usually described in the literature. The Marsden-Weinstein reduced space is denoted by $M_{\mathcal{O}}$ or by M_μ. The reduced space at the origin is denoted by $M_{//G}$.

Relation between M-W reduction and orbit reduction.

If we assume that the isotropy groups G_μ are connected for some, hence all, $\mu \in \mathcal{O}$, we can be more precise about the relation between B and $M_\mathcal{O}$: Let $b \in B$ and let $\mu = \Phi_B(b) \in \mathcal{O}$. Let $q \in \pi^{-1}(b)$. If $g \in G_\mu^0$ (the connected component of the isotropy subgroup of μ) then $\pi(gq) = b$ by our description of the null foliation of Q. We conclude that

$$G_\mu^0 \subset G_b.$$

On the the other hand, since Φ_B is a G-morphism, we know that $G_b \subset G_\mu$. So if G_μ is connected, we conclude that $G_b = G_\mu$, and hence that Φ_B gives a diffeomorphism of $G \cdot b$, the G orbit through b, with $\mathcal{O}$.

Consider $F_\mu := \Phi_M^{-1}(\mu)/G_\mu$ which we can identify with $M_\mathcal{O}$. The map

$$\gamma : \mathcal{O} \times F_\mu \to B, \quad \gamma([(a\mu, z)]) := az$$

is well defined. We claim that this is a diffeomorphism. Indeed we have the map

$$Q \to \Phi^{-1}_{\mathcal{O}^- \times M}(0), \quad m \mapsto (\Phi_M(m), m)$$

which induces a map of

$$\beta : Q/G \to \Phi^{-1}_{\mathcal{O}^- \times M}(0)/G = M_\mathcal{O}$$

and so the map

$$\Phi_M \times \beta : Q \to \mathcal{O} \times M_\mathcal{O}$$

which is the inverse of γ. $\square$

Elimination of the nodes.

Suppose $G = SO(3)$, the rotation group, acting in Hamiltonian fashion on (M, ω, Φ). Suppose that $0 \neq \mu \in \mathfrak{g}^*$ is a regular value. Then $\Phi^{-1}(\mu)$ has co-dimension 3 and the isotropy group of μ is just a circle. So the dimension of the reduced space M_μ is dim $M - 4$. The passage from M to M_μ is known as **elimination of the nodes** and was introduced by Jacobi in his study of the three body problem.

Reduction in stages.

Suppose that we have a group H also acting on the Hamiltonian G-space and commuting with the action of G. suppose that the action of H is also Hamiltonian. This means that we have a Hamiltonian action of $G \times H$ whose moment map is $\Phi_{G\times H} = \Phi_G \oplus \Phi_H$ under the natural identification of the dual of the Lie algebra of $G \times H$ with $\mathfrak{g}^* \oplus \mathfrak{h}^*$. Then H acts in a Hamiltonian fashion on $M_{\mathcal{O}}$ and, under the appropriate cleanness assumptions, if $\mathcal{P}$ is a coadjoint orbit for H, we have the natural identification of the double reduction

$$(M_{\mathcal{O}})_{\mathcal{P}}$$

with the single reduction (relative to $G \times H$)

$$M_{\mathcal{O}\times\mathcal{P}}.$$

See Meinrecken for details.

The *GxG* action on *G* .

Let G be a Lie group and let $G \times G$ act on G by left and right translation, so (a, b) acts on G by $L_a \circ R_b = R_b \circ L_a$. Even more explicitly (a, b) sends

$$c \mapsto acb^{-1}.$$

We can also write this as

$$L_{ab^{-1}} A_b$$

where A_b is conjugation by b.

The induced action on the cotangent bundle.

The action of $G \times G$ on G induces an action on T^*G with a moment map

$$T^*G \to \mathfrak{g}^* \oplus \mathfrak{g}^*.$$

We want to compute this moment map in the *left* identification of T^*G with $G \times \mathfrak{g}^*$. This means that we use dL_g to identify $\mathfrak{g} = T_eG$ with T_gG and hence $dL^*_{g^{-1}}$ to identify $\mathfrak{g}^*$ with T^*_gG. More explicitly, (g, μ) is identified with the element of T^*_gG which sends

$$(g, dL_gC) \mapsto \langle \mu, C \rangle, \quad C \in \mathfrak{g}.$$

The moment map for the right action.

Let us first consider the moment map for the right action of G on T^*G. If $C \in \mathfrak{g}$, the generating vector field for the right action is the vector field for the one parameter family of transformations

$$g \mapsto g \exp tC$$

which is just the vector field

$$C_G^R(g) = (g, C)$$

under our left identification of TG with $G \times \mathfrak{g}$. So the moment map for the right action is

$$\Phi^R : \; G \times \mathfrak{g}^* \to \mathfrak{g}^* \quad \Phi^R((g, \mu)) = \mu.$$

The moment map for the left action.

If $C \in \mathfrak{g}$, the generating vector field for the left action is the vector field for the one parameter family of transformations

$$g \mapsto \exp(-tC)g = g \cdot g^{-1}\exp(-tC)g.$$

Under the left identification of TG with $G \times \mathfrak{g}$ the value of this vector field at g is $(g, -\mathrm{Ad}_{g^{-1}} C)$. So the moment map for the left action is

$$\Phi^L(g, \mu) = -g \cdot \mu$$

where $g \cdot u$ denotes the coadjoint action of G on $\mu \in \mathfrak{g}^*$. To summarize:

Summary:

Theorem 1 *If we use the left identification of T^*G with $G \times \mathfrak{g}^*$ then the moment map for the $G \times G$ action on T^*G is*

$$\Phi^L \oplus \Phi^R : \; T^*G = G \times \mathfrak{g}^* \to \mathfrak{g}^* \oplus \mathfrak{g}^*$$

where

$$\Phi^L(g, \mu) = -g \cdot \mu$$

and

$$\Phi^R(g, \mu) = \mu.$$

The reproducing property of T^*G.

Let (M, ω, Φ_M) be a Hamiltonian G-space and consider the action of G on

$$(T^*G)^- \times M$$

where g acts to the right on T^*G. So the moment map for this action is

$$\Phi = \Phi_M - \Phi^R.$$

If we use the left identification of T^*G with $G \times \mathfrak{g}^*$ this becomes

$$\Phi(g, \mu, m) = \Phi_M(m) - \mu.$$

So $\Phi^{-1}(0)$ consist of all (g, μ, m) with $\mu = \Phi_M(m)$. Thus

$$\Phi^{-1}(0) = G \times M.$$

If we use the left identification of T^*G with $G \times \mathfrak{g}^*$ this becomes

$$\Phi(g, \mu, m) = \Phi_M(m) - \mu.$$

So $\Phi^{-1}(0)$ consist of all (g, μ, m) with $\mu = \Phi_M(m)$. Thus

$$\Phi^{-1}(0) = G \times M.$$

Dividing out by G gives M as a symplectic manifold. The left action of G on T^*G (with trivial action on M) commutes with the right action. So it restricts to an action on $\Phi^{-1}(0)$ which descends to a Hamiltonian action on $\Phi^{-1}(0)/G$ which is just the original action on M. In short.

$$\left(T^*G^- \times M\right)_{//G} = M. \tag{1}$$

Symplectic induction.

$$\left(T^*G^- \times M\right)_{//G} = M. \tag{1}$$

I want to generalize the construction involved in (1) in two ways. In this section I want to replace $G \times G$ by $G \times K$ where K is a Lie subgroup of G. If ι denotes the injection of $\mathfrak{k}$, the Lie algebra of K as a subalgebra of $\mathfrak{g}$ and if $\pi = \iota^*$ denotes the transpose map

$$\pi : \mathfrak{g}^* \to \mathfrak{k}^*$$

then the moment map for the right action of K on T^*G under the left identification of R^*G with $G \times \mathfrak{g}^*$ is

$$(g, \mu) \mapsto \pi\mu.$$

Now suppose that M is a Hamiltonian K-space with moment map

$$\Phi_M : M \to \mathfrak{k}^*.$$

Then $(T^*G)^- \times M$ is a Hamiltonian K space with moment map

$$\Psi : (T^*G)^- \times M \to \mathfrak{k}^*$$

where

$$\Psi(g, \mu, m) = \Phi_M(m) - \pi(\mu)$$

under our left identification of T^*G with $G \times \mathfrak{g}^*$. Then

$$\Psi^{-1}(0) = \{(g, \mu, m) | \pi(\mu) = \Phi_M(m)\}$$

and K acts freely on this set since it acts freely to the right on G. If K is compact, for example, we can then form the quotient space for this K action to obtain

$$\Big((T^*G)^- \times M\Big)_{//K}$$

$$\Psi^{-1}(0) = \{(g, \mu, m) | \pi(\mu) = \Phi_M(m)\}$$

and K acts freely on this set since it acts freely to the right on G. If K is compact, for example, we can then form the quotient space for this K action to obtain

$$\Big((T^*G)^- \times M\Big)_{//K}$$

which is a Hamiltonian G action derived from the left action of G on T^*G.

In this way, we have produced a Hamiltonian G space out of a Hamiltonian K-space. This is the symplectic version of induction, which, in representation theory produces a representation of a group out of a representation of a subgroup.

The moment map of a symplectic representation.

Suppose we are given a representation of K on a symplectic vector space (E, ω_E). I claim that this is Hamiltonian with moment map

$$\Phi_E;\ E \mapsto \mathfrak{k}^*, \quad \langle \Phi(v), A \rangle = \frac{1}{2}\omega_E(v, A \cdot v),\ v \in E,\ \ A \in \mathfrak{k}. \quad (2)$$

Indeed, under the vector space identification of $T_v E$ with E, the generating vector field for $A \in \mathfrak{k}$ is

$$A_E(v) = -A \cdot v.$$

So for any $w \in E$ we have

$$(i(A_E(v))\omega_E)(w) = \omega_E(A_E(v), w) = \omega_E(w, A \cdot v).$$

We must check that this coincides with the differential of the right right hand side of (2)

To prove: $$\langle \Phi(v), A\rangle = \frac{1}{2}\omega_E(v, A\cdot v),\ v\in E,\quad A\in\mathfrak{k}. \quad (2)$$

$$A_E(v) = -A\cdot v.$$

So for any $w \in E$ we have

$$(i(A_E(v))\omega_E)(w) = \omega_E(A_E(v), w) = \omega_E(w, A\cdot v).$$

We must check that this coincides with the differential of the right hand side of (2) when evaluated on w thought of as a tangent vector at v. But by definition, this is the derivative of $\frac{1}{2}\omega_E(v + tw, A\cdot(v+tw))$ evaluated at at $t = 0$. By Leibnitz this is

$$\frac{1}{2}\left[\omega_E(w, A\cdot v) + \omega_E(v, A\cdot w)\right] = \omega_E(w, A\cdot v)$$

as required, since A acts on E as an infinitesimal symplectic transformation.

The formula (2) is a generalization of the formula we obtained earlier for unitary representations.

The vector bundle structure on the induced space.

I claim that the reduced space

$$F := \left((T^*G)^- \times E\right)_{//K} = \Psi^{-1}(0)/K$$

has the structure of a homogenous G vector bundle over G/K. I will prove this under the assumption that $\mathfrak{k}^0 \subset \mathfrak{g}^*$ has a has a K-invariant complement, which is always true if K is compact. (Here $\mathfrak{k}^0$ denotes the space of those elements of $\mathfrak{g}^*$ which vanish when restricted to $\mathfrak{k}$.) We can then can then identify this complement with $\mathfrak{k}^*$. In other words, we have the direct sum decomposition

$$\mathfrak{g}^* = \mathfrak{k}^0 \oplus \mathfrak{k}^*.$$

The vector bundle structure on the induced space, 2.

$$\mathfrak{g}^* = \mathfrak{k}^0 \oplus \mathfrak{k}^*.$$

So if we write an element of $\mathfrak{g}^*$ as

$$\alpha + \beta, \quad \alpha \in \mathfrak{k}^0 \quad \beta \in \mathfrak{k}^*$$

we have

$$i^*(\alpha + \beta) = \beta.$$

Let us set

$$V := \mathfrak{k}^0 \oplus E.$$

The group K acts on $\mathfrak{k}^0$ by (the restriction of) the co-adjoint action, and on E via the given representation, and so acts on V.

The vector bundle structure on the induced space, 3.

Let us set

$$V := \mathfrak{k}^0 \oplus E.$$

The group K acts on $\mathfrak{k}^0$ by (the restriction of) the co-adjoint action, and on E via the given representation, and so acts on V. From the above we see that the map

$$G \times V \to \Psi^{-1}(0), \quad (b, \alpha, v) \mapsto (b, \alpha + \Phi_E(v), v)$$

is a K-equivariant identification. Hence the vector bundle $(G \times V)/K$ becomes identified with F.

The vector bundle structure on the induced space, 4.

Recall that we get a Hamiltonian action of G on F as follows: The left action of G on T^*G extends to an action on $(T^*G)^- \times E$ by letting G act trivially on E. The moment map for this action is

$$(b, \gamma, v) \mapsto b \cdot \gamma$$

so the restriction to $\Psi^{-1}(0)$ is

$$(b, \alpha + \Phi_E(v), v) \mapsto b \cdot (\alpha + \Phi_E(v)) \qquad (3)$$

which is K-eqivariant and so descends to F.

The symplectic normal bundle of G/K is (GxE)/K.

Notice that the homogeneous space G/K thought of as the zero section of the vector bundle F lies in the zero level set of this moment map and hence is isotropic as a submanifold of F. Its symplectic normal bundle is exactly the homogenous symplectic vector bundle $(G \times E)/K$.

A local model near an isotropic orbit.

$$(b, \alpha + \Phi_E(v), v) \mapsto b \cdot (\alpha + \Phi_E(v)) \tag{3}$$

We can use the construction of the last section and especially equation (3) to construct a *local model* for the moment map near an isotropic orbit. Here is how it works: Let $Y = G \cdot p$ be an isotropic orbit of a Hamiltonian G-action on a symplectic manifold (M, ω_M, Φ_M). Let $K = G_p$ so that $Y = G/K$ as a homogeneous space. The symplectic normal bundle of Y has as its fiber over p the space

$$E = I^{\perp}/I$$

where

$$I = TY_p = \mathfrak{g}/\mathfrak{k}$$

A local model near an isotropic orbit, 2.

$Y = G \cdot p \qquad K = G_p$ so that $Y = G/K \qquad I = TY_p = \mathfrak{g}/\mathfrak{k}$

$$TM_p = I^{\perp} \oplus C$$

where C is non-singularly paired with I under the symplectic form $\omega_M(p)$. In other words,

$$C = I^* = (\mathfrak{g}/\mathfrak{k})^* = \mathfrak{k}^0$$

as a K-space. Thus the $V = \mathfrak{k}^0 \oplus E$ constructed above can be identified with the fiber N_p of the normal bundle to $Y = G \cdot p = G/K$. This shows that the vector bundle $(G \times V)/K$ can be thought of as the ordinary normal bundle to Y.

A canonical form for the moment map near an isotropic orbit.

$$(b, \alpha + \Phi_E(v), v) \mapsto b \cdot (\alpha + \Phi_E(v)) \qquad (3)$$

But more to the point: we have two G-equivariant isotropic embeddings of G/K, as the zero section of F and as Y in M, and they have the same symplectic normal bundle, namely the bundle $(G \times E)/K$. Thus, by the isotropic embedding theorem, these are locally equivariantly symplectomorphic. In other words, the embedding of G/K as the zero section of F is a model for any isotropic orbit with the given symplectic normal bundle and (3) is a canonical form for the moment map in a neighborhood of the isotropic orbit.

In particular this applies when the orbit lies in the inverse image of a point in g^* which is fixed under the co-adjoint action of G.

Local convexity for Hamiltonian torus actions.

Let M be a Hamiltonian $\mathbf{T}$ space where $\mathbf{T}$ is a torus and let $\mathcal{O} = \mathbf{T} \cdot m$ be the orbit through a point $m \in M$. We know that this orbit is isotropic. So if $K = \mathbf{T}_m$ is the isotropy group of m we have the model

$$(T^*(\mathbf{T})^- \times E)//K$$

for the action of $\mathbf{T}$ near $\mathcal{O}$. We know that by appropriate choice of complex structure, the representation of K on E is unitary. Then (3) says that the image of the moment map for the action on this model is the cone

$$\mu + \pi^{-1}(\Phi_E(E))$$

Then (3) says that the image of the moment map for the action on this model is the cone

$$\mu + \pi^{-1}(\Phi_E(E))$$

where

$$\pi : \mathfrak{t}^* \to \mathfrak{k}^*$$

is the projection dual to the injection

$$\iota : \mathfrak{k} \to \mathfrak{t}$$

of the Lie algebra of K into the Lie algebra of $\mathbf{T}$.

Reverting back to the notation of the previous lecture, we have proved:

Local convexity for torus actions.

Theorem 2 *Let (M, ω, Φ_M) be a Hamiltonian $\mathbf{T}$ space where $\mathbf{T}$ is a torus. Then every $x \in M$ has a neighborhood O such that there is an open set O' in $\mathfrak{t}^*$ and a cone $C(x)$ in $\mathfrak{t}^*$ such that*

$$\Phi(O) = O' \cap C(x).$$

More precisely, the cone $C(x)$ is

$$C(x) := \Phi_m(x) + \left\{ \mu \in \mathfrak{t}^* \,\middle|\, \mu_{|\mathfrak{t}_x} = \sum r_i \beta_i, \;\; r_i \geq 0 \right\}$$

where the β_i are the weights of the representation of $\mathbf{T}_x^0$ on $T_x M$.