Homework #3 for Math 136

Due date: Wednesday Oct 13, 2004

From Hsiung's book:
Page 136: #3, #5, #6.
Page 138: #14.

Additional problems(Parallel curves): Let $\vec{X} : [a, b] \to \mathbb{R}^2$ be a arc-length parametrized regular plane curve with Frenet frame $\{\vec{e}_1(s), \vec{e}_2(s)\}$ and curvature $\kappa(s)$. For a fixed $r \in \mathbb{R}$, define a curve

$$\vec{X}_r(s) = \vec{X}(s) + r\vec{e}_2(s).$$

What is the curvature of the new curve $\vec{X}_r(s)$? What is the length of $\vec{X}_r(s)$(Note that $\vec{X}(s)$ is not necessary closed and convex, so the Cauchy formula will not apply)?

Bonus problem(Isoperimetric constant): Prove that there exists a constant $c > 0$ such that for any function $u(x) \in C_c^\infty(\mathbb{R}^2)$, (the set of all smooth functions with compact support in $\mathbb{R}^2$) we have

$$\left(\int_{\mathbb{R}^2} |u(x)|^2 dx\right)^{1/2} \le c \int_{\mathbb{R}^2} |\nabla u(x)| dx.$$

Remark: This inequality shows that the quantity

$$I_1(\mathbb{R}^2)^{1/2} = \inf_{u(x) \in C_c^\infty(\mathbb{R}^2)} \frac{\int_{\mathbb{R}^2} |\nabla u(x)| dx}{(\int_{\mathbb{R}^2} |u(x)|^2 dx)^{1/2}}$$

is strictly positive(a lower bound is given by $1/c$).