Final Exam for Math 136

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Due: 5pm Jan 14, 2005

It is an open book exam. You can use the textbooks and your notes. If you want to quote a result in one of the homework assignments, please state it clearly. The due date is 5pm Friday Jan 14. Please slide the exam in under my office door(SC 420).

None of the problems(except #1) requires messy computation. But some of them might be a little tricky. If you have any questions, please email me.

Good Luck!

1. A surface S is called *minimal* if the mean curvature at every point of S is equal to zero. Let S be a surface given by a graph of a function $f(x, y)$, ($(x, y) \in D \subset \mathbb{R}^2$). , i.e. S is parametrized by $\vec{X}(x, y) = (x, y, f(x, y))$.

Prove that S is minimal if and only if

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0,$$

where f_x denotes the partial derivative of f with respect to x, etc.

2. Let S be an embedded surface in $\mathbb{R}^3$ with the position vector $\vec{X}(p)$ and the unit outward normal vector $\vec{N}(p)$ for $p \in S$. For a fixed (small) t, define a surface S_t to be the set

$$S_t = \{\vec{X}(p) + t\vec{N}(p) \in \mathbb{R}^3 | p \in S\}.$$

Let κ_1, κ_2 be the principal curvatures of S at the point p with respect to the outward normal vector. Let H_t be the mean curvature of S_t at the point $\vec{X}(p) + t\vec{N}(p)$ with respect to the outward normal vector(Mean curvature is defined to be the sum of the two principal curvatures). Show that

$$H_t = \frac{\kappa_1}{1 - t\kappa_1} + \frac{\kappa_2}{1 - t\kappa_2}.$$

3. Let M be a connected compact(hence orientable) surface in $\mathbb{R}^3$ with positive Gaussian curvature. Prove that the Gauss map $\Gamma : M \to S^2$ is a differemorphism(S^2 is the unit sphere).

Hint: You can use the following topological result without proof: M is a compact surface in $\mathbb{R}^3$. $\varphi : M \to S^2$ is a regular map such that the tangent map $\varphi_{*p} : T_pM \to T_{\varphi(p)}S^2$ is injective for any $p \in M$. Then φ is a differeomorphism.

4. Let M be a compact surface in $\mathbb{R}^3$ with positive Gaussian curvature. Prove that for any $p \in M$, M lies entirely on one side of the tangent plane T_pM.

5.(a) Let M be a compact surface in $\mathbb{R}^3$. Show that there exists at least one point $p \in M$ such that the Gaussian curvature of M at p is positive(i.e. strictly bigger than zero).

(b) Prove that in $\mathbb{R}^3$, there does not exist compact minimal surfaces.

6. Let M be a surface in $\mathbb{R}^3$ and $p \in M$. $\sigma : M \to M$ is an isometry such that $\sigma(p) = p$ and $\sigma_{*p}\vec{v} = -\vec{v}$ for any $\vec{v} \in T_pM$. Let $\gamma : (-1, 1) \to M$ be a geodesic with $\gamma(0) = p$ and $\vec{X}(t)$ be a parallel field along γ. Prove that

$$\sigma_{*\gamma(t)}(\vec{X}(t)) = -\vec{X}(-t).$$

Remark: σ_{*q} denote the tangent map of σ at point q.