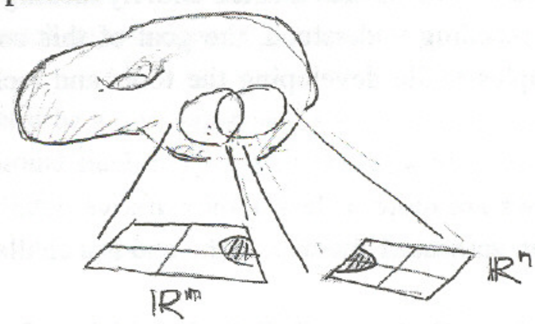
Math 135

Math 135 is a course on differential topolgy; thus, its focus is on the structure of smooth manifolds. This said, I need to tell you what smooth manifolds are. These are spaces that behave at small scales like Euclidean space, but are not so constrained at large scales. They can be very convoluted in the large, but quite regular in the small. To elaborate, each point in a manifold has some neighborhood that admits an identification with a Euclidean space. If the manifold is connected, then all of these Euclidean spaces have the same dimension, and the latter integer is then called the dimension of the manifold. Here is a schematic picture:



An identification between a given point neighborhood and a Euclidean space is viewed as a map from the former to the latter. The neighborhood of the point is called a 'coordinate chart' and any chosen basis for the linear functions on the Euclidean space is called a 'coordinate system'.

Now, here is a straightforward, but central observation: Where two of these neighborhoods intersect, you can follow the inverse of the first map and then the second map to get a map from an open region in Euclidean space to another open region in a copy of this same Euclidean space. The latter map is called a 'transition function', and it tells you how the coordinates in one region relate to those the intersecting region. By its very definition, such a transition function is continuous and also invertible with continuous inverse. It is therefore a homeomorphism.

A space of the sort just described is properly called a 'topological manifold', this to distinguish it from one whose coordinate chart transition functions have extra properties. Of prime interest in this course are the cases where all transition functions are differentiable to all orders. A manifold together with a collection of charts whose transition functions are infinitely differentiable is said to have a 'smooth structure' and is then called a 'smooth manifold'. In this regard, smooth manifolds are especially interesting because they are the natural arena for calculus. This is to say that the notion of derivatives to any order makes intrinsic sense on a smooth manifold. Calculus is impossible on a manifold that doesn't admit charts with differentiable transition functions for the simple reason that a function that might appear differentiable in one chart will not, in general, be differentiable when viewed from a neighboring chart. Here is a 1-

dimensional example: Suppose a transition function mapped a neighborhood of the origin in the real line to another neighborhood of the origin according to the rule $t \to \text{sign}(t)\,|t|^{1/3}$. Then the eminently differentiable function $f(t) = t$ would pull-back as the continuous but not so differentiable function $\text{sign}(t)\,|t|^{1/3}$.

With regard to this notion of a smooth structure, two different collections of charts with smooth transition functions are said to give the same smooth structure on some given topological manifold when all functions that are differentiable with respect to the coordinates from the first collection of charts are differentiable with respect to those of the other, and vice-versa. As it turns out a manifold may have more than one smooth structure. There are also manifolds that are entirely lacking in smooth structures.

With the preceding understood, the goal of this course is to investigate various fundamental examples while developing the tools and techniques to analyze any given case.

What follows are more or less administrative details that I want you to be aware of before we start in earnest. However, at the end is a challenge problem.

The Prof: I am Professor Cliff Taubes. My office is room 504 in the Science Center. My phone number is 5-5579 and email address is chtaubes@math. Office hours are: Tuesdays 1-2:30pm and Fridays 2-3:30pm.

Course Times: The course meets on Tuesdays and Thursdays from 11:30-1pm in Science Center room 304. A weekly, hour long review/problem session to be run by the Course Assistant will also be scheduled.

The Texts: Reading and homework problems will be from two books: *Differential Topology* by Guilleman and Pollack, and *Topology from the Differentiable Viewpoint* by John Milnor. These should be available at the Coop.

Grading: You will be graded on one of the following two schemes:

- Scheme 1: Homework scores count 30%, a midterm project counts 30% and a final project counts 40%.
- Scheme 2: Homework scores count 20%, a midterm project counts 20% and a final project counts 60%.

An irrevocable choice must be made by each student prior to the distribution of the final project. Those not choosing will be graded according to Scheme 1.

Homework: Homework problems will be assigned more or less weekly and due back in class one week later. Permission must be obtained from me to hand in a late assignment. However, you are allowed to miss one assignment during the term without incurring a grade penalty.

You are also free to collaborate with your classmates on homework assignments. Even so, you must write up your own assignment, and must list your collaborators on any given assignment.

Supplementary Topics: If you find that you are familiar with a given portion of this course, I can supply you with alternative reading and homework assignments. Let me know whenever you desire such supplementary work.

Challenge Problem: Let n be a non-negative integer. The sphere of dimension n is the subset of the n+1 dimensional Euclidean space where the coordinates, $(x_1, \ldots, x_{n+1})$, obey the rule $x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$.

- Find coordinate charts on such a sphere to prove that it has the structure of a smooth manifold.
- Prove that the spheres of dimensions 0 and 1 have but a single smooth structure.

(It is known that spheres of dimension n = 0, 1, 2, 5, 6 have just one smooth structure. Spheres of dimension 7 may more than one as do various n > 7 cases. For example, there are 28 such structures on the 7 dimensional sphere, 2 on the eight dimensional sphere, 8 on the 9 dimensional spherer, 6 on the ten dimensional sphere and 992 on the eleven dimensional sphere. For each $n \geq 5$ case, the precise number can be determined. On the other hand, no one knows whether the 4-dimensional sphere has more than one smooth structure. The original 'Poincare´ conjecture' asked whether the 3-dimensional sphere has just one smooth structure. A proof of this conjecture was submitted recently by Grisha Perelman and is now being evaluated by 'the experts'.)