Thursday May 1: Last class.

Final: online 9am on Sat
Extra Office Hours: Friday 1-4

write up a handout of this.

Thm: A connected compact 2-manifold is homeomorphic to exactly one of:

(orient): $S^2 =$, $T^2 =$, $T^2 \# T^2 =$, ..., $\underbrace{T^2 \# \cdots \# T^2}_{n \text{ summands } = \text{ surface of genus } n}$, ...

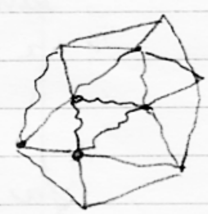
(non orient): $P^2 = \mathbb{R}P^2 =$, $P^2 \# P^2 =$, $P^2 \# P^2 \# P^2$, ...

Last time: Wild embeddings, etc.

Solution: work w/ a restricted class of maps: smooth maps

for triangulated surface. → piecewise linear

Thm: X a compact surface. Then X has a triangulation.
i.e. $\exists$ a finite # of embeddings $f_i : \Delta \to X$ whose images cover X and so that $f_i(\Delta) \cap f_j(\Delta) = \emptyset$, a vertex of both, an edge of both.

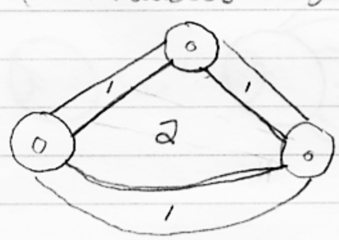


Pf: See Refs in Munkres 472.

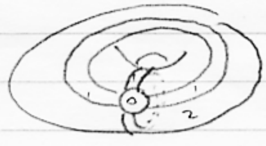
[mention usual proof]

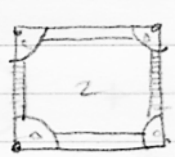
Handle Decompositions of 2-mflds:

$X = \bigcup$ (0-handles $= B^2$) $\cup$ (1-handles $= I \times I$ w/ $\partial I \times I$ glued to 0-handles) $\cup$ (2-handles $= B^2$ glued to 0 + 1 handles by all of ∂B^2)



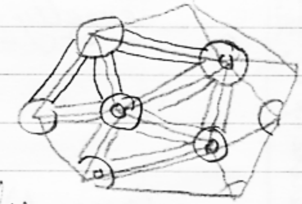
Ex:





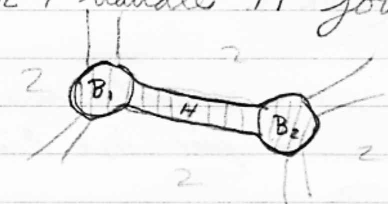
<u>Thm</u>: X conn. cpt surface. Then X has a handle decomp with only one 0-handle and one 2-handle.

<u>Pf</u>: Since X has a triang. it has some handle decomp



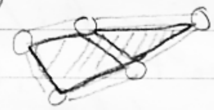
- Suppose X has a handle decomp w/ $n \geq 2$ 0-handles. Then X has one w/ $n-1$ 0-handles.

Pick 1-handle H joining two distinct 0-handles B_1 and B_2

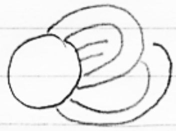


Let $H \cup B_1 \cup B_2$ be a 0-handle
Now the handle decomp has $n-1$ 0-handles

Similarly, can amalgamate 2 distinct 2-handles across a one handle.



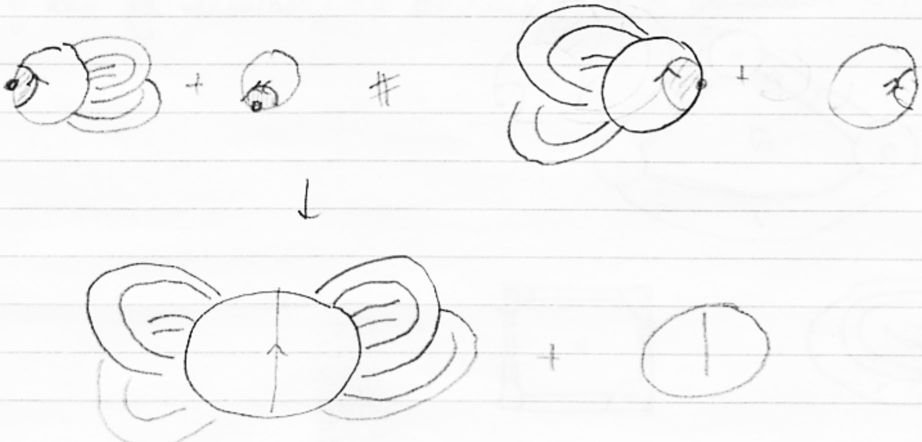
for such decomp, only need to remember 0 and 1 handles.

 + 2 handle = T^2

Moreover, only two types of 1-handle

only in non-orient mflds.

<u>Connected Sum</u>: $T^2 \# T^2$



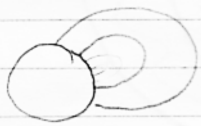
Pf of Classification Thm (orient case)

Part 1: every orient surface is S^2 or $T^2 \# \cdots \# T^2$

Consider a handle decomp of X w/ one 0-handle and one 2-handle and n - two handles. Claim $X \cong \underbrace{T^2 \# \cdots \# T^2}_{n/2 \text{ times}}$

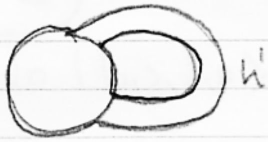
$n = 0 \Rightarrow X = S^2$ ✓

$n = 1 \Rightarrow$  not allowed as only have one 2-handle

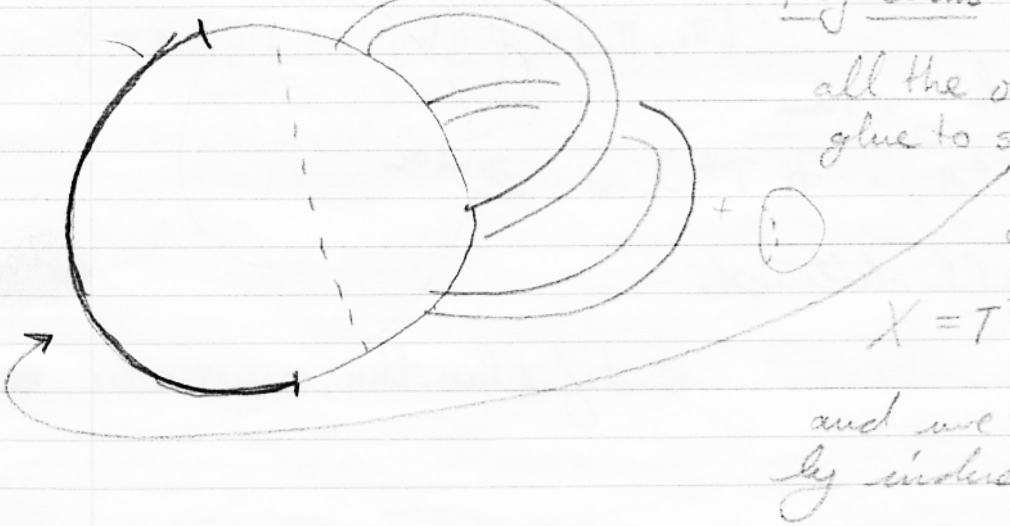
$n = 2 \Rightarrow$  + 1-handle + 2-handle. 2nd one handle must have one end glued to  and one to 

i.e.  ✓

General case: Let $h^1, \ldots, h^n$ be the two handles.



As before, there must exist some h^i going from one bdry comp to the other. reordering $h^i = h^2$



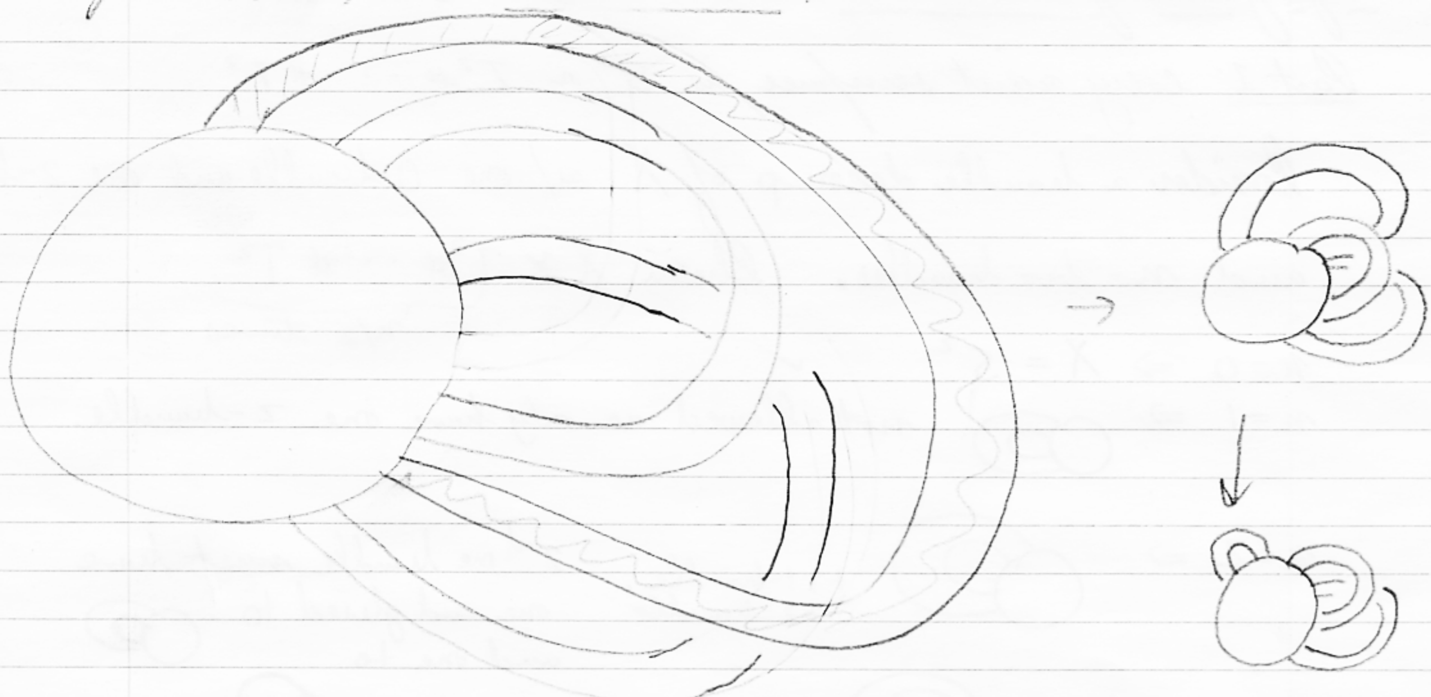
Key claim: can assume all the other h^i, $i > 2$ glue to section marked

If so,

$X = T^2 \# (\text{something w/ } n-2 \text{ 1-handles})$

and we'd be done by induction.

To prove claim, use <u>handleslides</u>



This proves the claim and completes part 1. ✓

<u>Part 2:</u> all the orient surfaces listed are distinct

$S^2 =$  $T^2 =$  $T^2 \# T^2$ 

$\pi_1 =$	1	$\mathbb{Z}\times\mathbb{Z}$	$\langle a,b,c,d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1}\rangle$
$\pi_1^{ab} =$	0	$\mathbb{Z}\times\mathbb{Z}$	$\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$

Abelianization: $\pi_1^{ab} = \pi_1 / [\pi_1, \pi_1] = \{ghg^{-1}h^{-1} \mid g,h \in \pi_1\}$

In general

$$\pi_1^{ab}(\overbrace{T^2 \# \cdots \# T^2}^{n \text{ times}}) \text{ is } \mathbb{Z}^{2n}$$

So they're all distinct.

End of class thm, orient case ■