Final for Math 131, Spring 2002

Deadline: The exam is due in my office (SciCen 334) by 3:00pm on Thursday, May 8. If you come by early when I'm not there, you can slip it under my door.
Disclaimer, Terms, and Conditions: You may not discuss the exam with anyone except myself. You may *only* consult the following:

- The beloved text (Munkres).
- Hatcher's *Algebraic Topology* (available online and on the math library reserve shelf).
- Your class notes.
- Old HW sets and solution handouts.

You may also use a calculator or computer to ease your labors, and consult non-topology math texts (e.g. on algebra or basic analysis) as seems appropriate.

While I believe all the questions are stated correctly, there could still be a typo somewhere. Please contact me if you think something is fishy.

Good luck and have fun!

Nathan

Actual exam: Do all six problems; all questions are weighted equally.

1. Recall that an n-manifold is a Hausdorff, second-countable, topological space X such that each point $x \in X$ has a neighborhood which is homeomorphic to $\mathbb{R}^n$.

   (a) Let X be an n-manifold. Prove that X is metrizable.

   (b) Again, let X be an n-manifold. Suppose X is connected. Need X be path connected?

2. A space X is *locally compact at x* if there is a neighborhood U of x and a compact set $C \subset X$ such that $U \subset C$. The space X is *locally compact* if it is locally compact at every point.

   Determine and prove whether I^ω is locally compact when I^ω is given

   (a) the product topology.

   (b) the uniform topology.

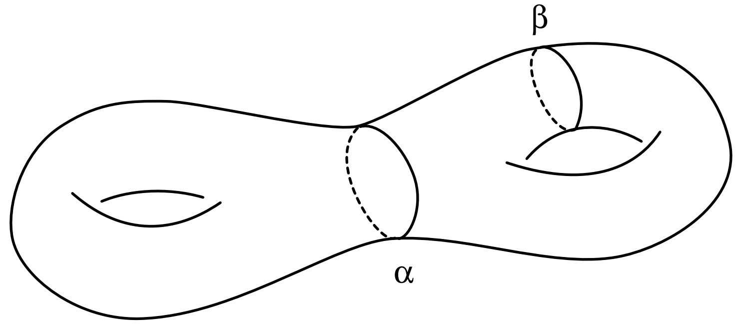
   (c) the box topology.

3. Let C be the standard middle-third Cantor set.

   (a) Let X be a compact metric space. Prove that there exists a continuous *surjection* from C onto X. (This is an important step in proving the Hahn-Mazurkiewicz theorem that a compact metric space which is connected and weakly-locally connected is the continuous image of an interval).

   (b) Now suppose instead that X is a compact Hausdorff topological space. Need there still exist a continuous surjection from C onto X?

4. Let X be the surface shown below and let α and β be the two indicated circles on X.



   (a) Prove that X does *not* retract onto α. Hint: Show that a torus with a disc removed does not retract to its boundary circle; that is, show that left half, Y, of X does not retract onto α. For this, use that a loop going once around α is in the commutator subgroup of $\pi_1(Y)$.

   (b) Prove that X does retract onto β.

5. Let K be the Klein bottle.

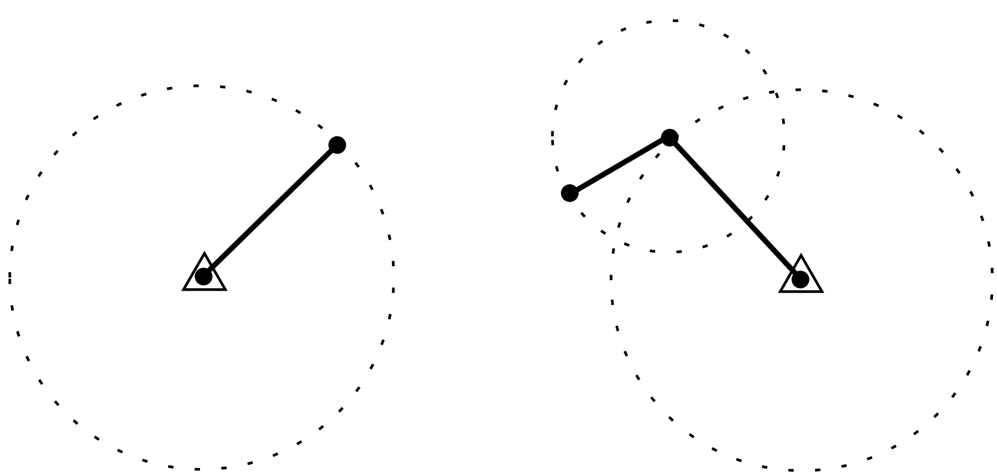
   (a) Find a presentation for $\pi_1(K)$.

   (b) Exhibit non-trivial regular covers of K by the torus and by K itself. Give generators of the corresponding (normal) subgroups of $\pi_1(K)$ in both cases.

   (c) Now exhibit an *irregular* cover $p\colon E \to K$ and give generators of the corresponding subgroup. What surface is E?

   (d) Prove that $\pi_1(K)$ is infinite and non-abelian.

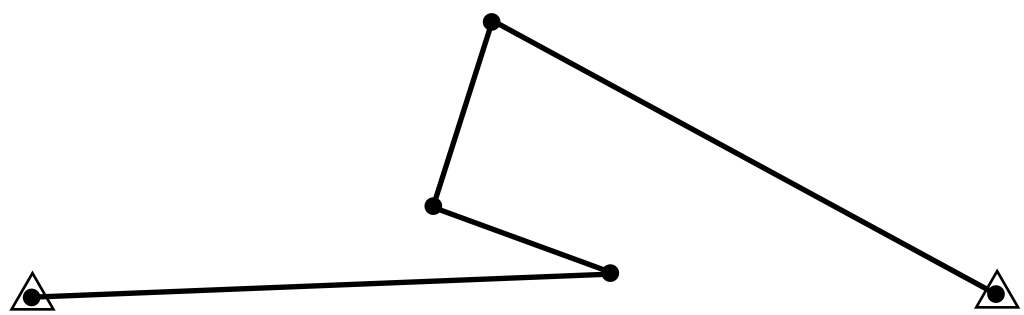
6. Manifolds often arise as the space of configurations of certain objects in $\mathbb{R}^n$, or as the set of solutions to some bunch of equations. Here is one such example. A *planar linkage* is basically a rigid set of rods, lying on tabletop, where the endpoints of the rods are attached together by freely-moving pivots and some endpoints are tacked to the plane by freely-moving pivots. Below are two sample planar linkages, where the triangles denote pivots that are attached to the plane.



We'll be interested in the configuration space $C(L)$ of a given linkage L; that is, the set of all possible configurations of L together with the natural metric that two configurations are close if all the endpoints of the rods are close in $\mathbb{R}^2$. In the examples, the configuration space of the linkage on the left is S^1, and the configuration space of the linkage at right is $T = S^1 \times S^1$.

To be more formal about it, you can make $C(L)$ a subset of $(\mathbb{R}^2)^n$ where the factors correspond to the pivots/endpoints of the various rods; then, $C(L)$ is just the solution space to the system of polynomial equations which say that the distance between a pair of pivots jointed by a rod is equal to the length of that rod.

Ok, now for the actual problem. Consider the linkage shown below.



Here the two triangles are 5 units apart, whereas the lengths of the two longer rods are 3 units and the lengths of the two shorter rods are 1 unit.

(a) Write down polynomial equations in 6 variables whose solution space is $C(L)$, using the positions of the 3 free pivots.

(b) In this case $C(L)$ is a compact, connected surface. Figure out which one. Hint: Project $C(L)$ onto the position of the middle free pivot.

Planar linkages are closely related to robot arms where both topology and algebraic geometry have real-world applications!

Extra Credit. For extra credit worth as half has much as one of the regular problems, try one of the questions below. If you manage to do more than one of them, I'll give you the maximum score over all your answers.

1. Construct a space X which is countable, Hausdorff, and connected.

2. Suppose X is a path connected space which is the union of two path connected closed sets A and B such that $A \cap B$ consists of a single point. Can we have $\pi_1(A) = \pi_1(B) = 1$ but $\pi_1(X) \neq 1$?

3. A theorem of W. Thurston says that any smooth n-manifold is a connected component of the configuration space of some linkage. Find a linkage L with $C(L)$ homeomorphic to the projective plane.