Homework 7: DIFFERENTS

Questions marked with an * are optional, i.e. not for credit.

1) Suppose that $K/\mathbf{Q}_p$ is a finite extension and let $\overline{K}$ denote the algebraic closure of K. Let C denote a positive number.

(a) Show that $K^\times/(K^\times)^p$ is finite. Deduce that K has only finitely many cyclic extensions of degree p in $\overline{K}$. [Hint: treat first the case that K contains a primitive p^{th} root of unity.]

(b) Show that K has only finitely many totally wildly ramified Galois extensions of degree $\leq C$ in $\overline{K}$.

(c) Show that K has only finitely many tamely ramified Galois extensions of degree $\leq C$ in $\overline{K}$.

(d) Show that K has only finitely many extensions of degree $\leq C$ in $\overline{K}$.

2) (a) Show that every quadratic extension of $\mathbf{Q}_2$ is contained in $\mathbf{Q}_2(\sqrt{-3}, \sqrt{-1}, \sqrt{2})$ and hence in $\mathbf{Q}_2(\zeta_{24})$, where ζ_{24} is a primitive 24^{th} root of unity.

(b) Show that $X^2 + Y^2 + Z^2$ has no non-trivial solution in $\mathbf{Q}_2^3$. [Hint: if there were such a solution one could assume that all the variables lie in $\mathbf{Z}_2$ and one is a unit.]

(c) Suppose that $L/\mathbf{Q}_2$ is a cyclic extension of degree 4 containing $\sqrt{-1}$, so that $L = \mathbf{Q}_2(\sqrt{-1})(\alpha)$, where

$$\alpha^2 = x + y\sqrt{-1}$$

with $x, y \in \mathbf{Q}_2$. Let σ be a generator of $\mathrm{Gal}(L/\mathbf{Q}_2)$. Show that

$$\sigma(\alpha(\sigma\alpha)) = -(\alpha(\sigma\alpha)),$$

and deduce that

$$\alpha(\sigma\alpha) = z\sqrt{-1}$$

for some $z \in \mathbf{Q}_2$. Then show that

$$x^2 + y^2 + z^2 = 0$$

and deduce that no such extension $L/\mathbf{Q}_2$ can exist.

(d) Show that every abelian extension of $\mathbf{Q}_2$ is contained in a cyclotomic extension.

3) Suppose that K is complete with respect to a non-trivial, non-archimedean absolute value $|\ |$. By a *fractional ideal* I of $\mathcal{O}_K$ we mean an $\mathcal{O}_K$-submodule $I \subset K$ which neither equals K nor (0).

(a) Show that if $I \neq (0)$ is an $\mathcal{O}_K$-submodule of K then I is a fractional ideal if and only if $|\ |$ is bounded on I.

(b) If I and J are fractional ideals of $\mathcal{O}_K$ let

$$I^{-1} = \{\alpha \in K : \alpha I \subset \mathcal{O}_K\}$$

and let IJ denote the $\mathcal{O}_K$ module generated by the set of elements $\alpha\beta$ with $\alpha \in I$ and $\beta \in J$. Show that I^{-1} and IJ are fractional ideals of $\mathcal{O}_K$.

(b) If $r \in \mathbf{R}_{>0}$ let $I_{<r}$ (resp. $I_{\leq r}$) denote the set of elements $\alpha \in K$ with $|\alpha| < r$ (resp. $\leq r$). Show that $I_{<r}$ and $I_{\leq r}$ are fractional ideals of $\mathcal{O}_K$. If $|\ |$ is discrete show that every fractional ideal is of the form $I_{\leq r}$ with $r \in |K^\times|$ and that these fractional ideals are all distinct. If $|\ |$ is not discrete show that every fractional ideal is of the form $I_{<r}$ for $r \in \mathbf{R}_{>0}$ or $I_{\leq r}$ for $r \in |K^\times|$ and that these fractional ideals are all distinct.

(c) If $|\ |$ is discrete show that every fractional ideal I is principal, i.e. $I = \mathcal{O}_K\alpha$ for some $\alpha \in K^\times$, and that $II^{-1} = \mathcal{O}_K$. If $|\ |$ is not discrete show that some ideals I of $\mathcal{O}_K$ are not finitely generated and do not satisfy $II^{-1} = \mathcal{O}_K$.

(d) If $r, s \in |K^\times|$ show that

$$I_{\leq r}^{-1} = I_{\leq r^{-1}}$$

and that

$$I_{\leq r} I_{\leq s} = I_{\leq rs}.$$

(e) If $|\ |$ is not discrete, if $r \in |K^\times|$ and if $s, t \in \mathbf{R}_{>0}$ show that

$$I_{\leq r} I_{<s} = I_{<rs}$$

and

$$I_{<s} I_{<t} = I_{<st}$$

and

$$I_{<r}^{-1} = I_{\leq r^{-1}}.$$

If $s \notin |K^\times|$ show further that

$$I_{<s}^{-1} = I_{<s^{-1}}.$$

(f) If I is a fractional ideal and $\alpha \in K$ satisfies $\alpha I \subset I$ show that $\alpha \in \mathcal{O}_K$. If $|\ |$ is discrete and if I and J are fractional ideals of $\mathcal{O}_L$ show that

$$\{\alpha \in K : \ \alpha I \subset J\} = I^{-1}J.$$

(g) If L/K is a finite extension and I is a fractional ideal of $\mathcal{O}_L$ we define its *norm*, $\mathrm{N}_{L/K} I$, to be the $\mathcal{O}_K$-submodule of K generated by the set of $\mathrm{N}_{L/K}(\alpha)$ with $\alpha \in I$. Show that $\mathrm{N}_{L/K} I$ is a fractional ideal of $\mathcal{O}_K$. More specifically show that if $r \in |L^\times|$

$$\mathrm{N}_{L/K} I_{\leq r} = I_{\leq r^{[L:K]}};$$

while if $|\ |$ is not discrete and $s \in \mathbf{R}_{>0}$ then

$$\mathrm{N}_{L/K} I_{<s} = I_{<s^{[L:K]}}.$$

Also show that

$$\mathrm{N}_{L/K}(IJ) = (\mathrm{N}_{L/K} I)(\mathrm{N}_{L/K} J)$$

and

$$\mathrm{N}_{L/K}(I^{-1}) = (\mathrm{N}_{L/K}I)^{-1}.$$

If I is a fractional ideal of $\mathcal{O}_K$ show that

$$\mathrm{N}_{L/K}(I\mathcal{O}_L) = I^{[L:K]}.$$

Finally show that if M/L is also a finite extension then

$$\mathrm{N}_{L/K}\mathrm{N}_{M/L}I = \mathrm{N}_{M/K}I.$$

4) Suppose that K is complete with respect to a non-trivial, non-archimedean absolute value $|\ |$ and that L/K is a finite separable extension. Define the *inverse different*

$$\mathcal{D}_{L/K}^{-1} = \{\alpha \in L : \ \mathrm{tr}_{L/K}(\alpha\mathcal{O}_L) \subset \mathcal{O}_K\}.$$

Also define the *different*

$$\mathcal{D}_{L/K} = (\mathcal{D}_{L/K}^{-1})^{-1}$$

and the *discriminant*

$$D_{L/K} = \mathrm{N}_{L/K}\mathcal{D}_{L/K}.$$

(a) Show that $\mathcal{D}_{L/K}^{-1}$ and $\mathcal{D}_{L/K}$ and $D_{L/K}$ are fractional ideals. Show also that $\mathcal{D}_{L/K}^{-1} \supset \mathcal{O}_L$, that $\mathcal{D}_{L/K} \subset \mathcal{O}_L$ and that $D_{L/K} \subset \mathcal{O}_K$.

(b) Show that the $\mathcal{O}_K$-bilinear pairing

$$\begin{array}{rcl} \mathcal{D}_{L/K}^{-1} \times \mathcal{O}_L & \longrightarrow & \mathcal{O}_K \\ (\alpha,\beta) & \longmapsto & \mathrm{tr}_{L/K}(\alpha\beta) \end{array}$$

is a perfect pairing, i.e. induces isomorphisms

$$\mathcal{D}_{L/K}^{-1} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_K)$$

and

$$\mathcal{O}_L \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{D}_{L/K}^{-1}, \mathcal{O}_K).$$

(c) Show that $\mathcal{D}_{L/K}^{-1}$ is a principal fractional ideal.

(d) If $|\ |$ is discrete and if I is a fractional ideal of $\mathcal{O}_L$ show that

$$\begin{array}{rcl} \mathcal{D}_{L/K}^{-1}I^{-1} \times I & \longrightarrow & \mathcal{O}_K \\ (\alpha,\beta) & \longmapsto & \mathrm{tr}_{L/K}(\alpha\beta) \end{array}$$

is a perfect pairing.

(e) If M/L is finite and separable, show that

$$\mathcal{D}_{M/K}^{-1} = \mathcal{D}_{L/K}^{-1}\mathcal{D}_{M/L}^{-1}$$

and that

$$D_{M/K} = D_{L/K}^{[M:L]}(\mathrm{N}_{L/K}D_{M/L}).$$

(f) If L/K is unramified and if $\overline{e}_1, ..., \overline{e}_r$ is a basis of $k(L)$ over $k(K)$ show that they can be lifted to a basis $e_1, ..., e_r$ of $\mathcal{O}_L$ over $\mathcal{O}_K$. [Hint: first show that $e_1, ..., e_r$ span L as a K-vector space and then show that if $\alpha_1, ..., \alpha_r \in K$ then show that $|\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_r e_r|_L$ equals the maximum of the $|\alpha_i|$.] Deduce that $\mathcal{D}_{L/K}^{-1} = \mathcal{O}_L$.

(g) If $|\ |$ is discrete show that

$$\mathcal{D}_{L/K} \subset \varpi_L^{e_{L/K}-1}\mathcal{O}_L.$$

In this case deduce that L/K is unramified if and only if $D_{L/K} = \mathcal{O}_K$.

5) Suppose that K is complete with respect to a non-trivial, non-archimedean absolute value $|\ |$ and that L/K is a finite separable extension. Suppose also that $|\ |$ is discrete and that the residue field $k(K)$ of K is perfect.

(a) Show that there is an element $\beta \in \mathcal{O}_L$ such that

$$\mathcal{O}_L = \mathcal{O}_K[\beta].$$

[Hint: Let ϖ_L be a uniformizer for $\mathcal{O}_L$, let $k(L) = k(K)(\overline{\alpha})$, let $\overline{f}(T)$ denote a monic minimal polynomial for $\overline{\alpha}$ over $k(K)$ and let f denote a monic lift of $\overline{f}$ to $\mathcal{O}_K[T]$. Explain why there is an element α of $\mathcal{O}_L$ lifting $\overline{\alpha}$ with $f(\alpha) = 0$. Let $\beta = \alpha + \varpi_L$. Show that

$$|f(\beta)| = |\varpi_L|.$$

Deduce that $\mathcal{O}_L = \mathcal{O}_K[\beta]$.]

(b) Let g denote the monic minimal polynomial of β over K. Also show that

$$\mathcal{D}_{L/K} = g'(\beta)\mathcal{O}_L.$$

[Hint: Let $\beta = \beta_1, ..., \beta_n$ denote the roots of g. Show that

$$1/g(T) = \sum_{i=1}^{n} 1/(g'(\beta_i)(T - \beta_i)).$$

Deduce that

$$T^{-n}\sum_{i=0}^{\infty}(1 - T^{-n}g(T))^i = \sum_{j=1}^{\infty} T^{-j}\mathrm{tr}_{L/K}(\beta^{j-1}/g'(\beta))$$

in $K[[T^{-1}]]$, and hence that

$$\mathrm{tr}_{L/K}(\beta^i/g'(\beta)) \begin{cases} = 0 & \text{for } \ 0 \le i \le n-2 \\ = 1 & \text{for } \ i = n-1 \\ \in \mathcal{O}_K & \text{for } \ i > n-1. \end{cases}$$

Conclude that $\beta^i/g'(\beta)$ for $i = 0, ..., n-1$ are an $\mathcal{O}_K$ basis for $\mathcal{D}_{L/K}^{-1}$.]

6) (a) Let $K = \mathbf{Q}_2(\sqrt{5}, \sqrt{2})$. Find $\mathcal{O}_K$ and $\mathcal{D}_{K/\mathbf{Q}_2}^{-1}$.
(b) Let $K = \mathbf{Q}_2(\sqrt{-1}, \sqrt{2})$. Find $\mathcal{O}_K$ and $\mathcal{D}_{K/\mathbf{Q}_2}^{-1}$.
(c*) Let $L = \mathbf{Q}_2(\beta)$ where $(\beta^2 - 1)^2 = 2$. Find $\mathcal{O}_L$ and $\mathcal{D}_{L/\mathbf{Q}_2}^{-1}$.
[Hint: see Homework 6.]