Homework 6: RAMIFICATION

Questions marked with an * are optional, i.e. not for credit.

1) Suppose that K is complete with respect to a nontrivial valuation v with corresponding absolute value $|\ | = |\ |_{v,q}$. Suppose also that M/K is an *infinite* Galois extension. (For example K could equal $\mathbf{Q}_p$ and M could equal its algebraic closure.) In this question we will show that M is not itself complete (with respect to the unique extension $|\ |_M$ of $|\ |$). We will suppose that it is complete and derive a contradiction.

Show that we can find an infinite series of fields

$$K \subset K(\alpha_1) \subset K(\alpha_2) \subset K(\alpha_3) \subset ... \subset M$$

such that $K(\alpha_i)/K$ is Galois and $K(\alpha_{i+1}) \neq K(\alpha_i)$ for all i. Show further that we may assume that $|\alpha_i|_M \leq 1$. Choose $0 < \eta < 1$ and elements $\beta_i \in K$ such that $|\beta_i| < \eta$ and

$$|\beta_i| < |\sigma\alpha_i - \alpha_i|_M$$

for all $1 \neq \sigma \in \mathrm{Gal}(K(\alpha_i)/K)$. Show that

$$\alpha_1 + \beta_1\alpha_2 + \beta_1\beta_2\alpha_3 + ... + \beta_1...\beta_{i-1}\alpha_i + ...$$

converges to an element $\gamma \in M$. Suppose that σ is an automorphism of M which fixes every element of $K(\gamma)$. Show by induction on i that σ also fixes α_i for all i. [Hint: consider $\sigma\gamma - \sigma$.] Deduce that $K(\gamma)$ contains every α_i and hence obtain a contradiction.

2) Write down a polynomial over $\mathbf{Q}_3$ whose splitting field is the unramified extension of $\mathbf{Q}_3$ of degree 3.

3) Let $K = \mathbf{Q}_2[\sqrt{5}, \sqrt{2}]$. Find a uniformiser for K. Find the maximal unramified sub-extension of $K/\mathbf{Q}_2$. For all $i \in \mathbf{Z}_{>0}$ find $\mathrm{Gal}(K/\mathbf{Q}_2)_i$.

4) Let $K = \mathbf{Q}_3(\alpha)$ where $(2 - \alpha^2)^3 = 3$. Find a field $K \supset M \supset \mathbf{Q}_3$ with $M/\mathbf{Q}_3$ unramified of degree 2. Show that K/M is totally ramified.

5) Let $K = \mathbf{Q}_2[\sqrt{-1}, \sqrt{2}]$. Show that $\pi = (1 + \sqrt{-1} + \sqrt{2})/\sqrt{2}$ is a uniformizer for K. For all $i \in \mathbf{Z}_{>0}$ find $\mathrm{Gal}(K/\mathbf{Q}_2)_i$.

6) Let $L = \mathbf{Q}_2(\beta)$ where $(\beta^2 - 1)^2 = 2$. Show that $L/\mathbf{Q}_2$ is completely ramified of degree 4 and that $\mathcal{O}_L = \mathbf{Z}_2[\beta]$.

Show that the normal closure N of $L/\mathbf{Q}_2$ equals $L(\sqrt{-1})$. Also show that $\sqrt{-1} \notin L$. Describe the Galois group of $N/\mathbf{Q}_2$.

Find a uniformizer for N. [Hint: look for an element of N whose norm down to $\mathbf{Q}_2[\sqrt{-1}, \sqrt{2}]$ has the same valuation as an odd power of the element π in question 4).] Find an integer i such that $\mathrm{Gal}(N/\mathbf{Q}_2)_i = \mathrm{Gal}(N/\mathbf{Q}_2(\sqrt{-1}, \sqrt{2})$.

*For all $i \in \mathbf{Z}_{>0}$ find $\mathrm{Gal}(N/\mathbf{Q}_2)_i$.

7) Classify the finite groups which can occur as the Galois group of a finite Galois extension of $\mathbf{R}((T))$.